MATH 203, PROBLEM SET 3

DUE IN ASHER AUEL’S MAILBOX BY 5 P.M., MONDAY, FEB. 6.

Note: The mid-term exam will be given in class on Feb. 6.

1. Show that the following sets have the same cardinality:
   a. The interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$.
   b. The half open interval $(0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$.
   c. The positive real numbers $\mathbb{R}_{>0} = \{x \in \mathbb{R} : 0 < x\}$.
   d. The set $\mathbb{R}$ of all real numbers.

Hints: You can use the Schröder Bernstein Theorem, which says that two sets $A$ and $B$ have the same cardinality if there are injections $A \rightarrow B$ and $B \rightarrow A$. You can also use the usual algebra of the real numbers to establish these injections. So, for example, try writing down a simple formula for a function $f$ which is an injection $f : (0, 1] \rightarrow (0, 1)$. Notice that the function $g(x) = 1/(1 + x)$ sends $\mathbb{R}_{>0}$ to $(0, 1)$; is this injective? Finally, in getting from (c) to (d), notice that there is an bijection from $(0, 1) \cup [1, 2)$ to $(0, 1)$ defined by $x \rightarrow x/2$. How can you use this to get an injection from $\mathbb{R} = \mathbb{R}_{>0} \cup \mathbb{R}_{\leq 0}$ into $\mathbb{R}_{>0}$?

2. Suppose that there were a set $S$ which contains as its elements all possible sets. Show that if $2^S$ is the power set of $S$, then $\text{Card}(S) \leq \text{Card}(2^S)$ and $\text{Card}(2^S) \leq \text{Card}(S)$. What does this contradict?

3. Show that the set $S$ of all functions $f : \mathbb{Q} \rightarrow \mathbb{R}$ from the rational numbers $\mathbb{Q}$ to the real numbers $\mathbb{R}$ has the same cardinality as the set $S'$ of all functions $h : \mathbb{Z} \rightarrow \mathbb{R}$.

Extra Credit: Show that $S$ (and hence $S'$) has the same cardinality as $\mathbb{R}$.

Hints: You can use without proof the result proved in the last homework set that $\mathbb{Q}$ has the same cardinality as the integers $\mathbb{Z}$. For the extra credit part of the problem, one approach is to let $W$ be the set of all sequences $a = (a_1, a_2, a_3, \ldots)$ of real numbers $a_j$ in the interval $(0, 1)$ which are indexed by the set $\mathbb{Z}^+ = \{1, 2, \ldots \}$ of positive integers. Use problem 1 above and the fact that $\mathbb{Z}$ has the same cardinality as $\mathbb{Z}^+$ to show that $S'$ has the same cardinality as $W$. Show that the map $a_1 \rightarrow (a_1, 0, 0, 0, \ldots)$ is an injection $(0, 1) \rightarrow W$. Thus by the Schröder Bernstein Theorem, $(0, 1)$ and $W$ have the same cardinality if we can produce an injection $H : W \rightarrow (0, 1)$. Try to do this using the decimal expansions

$$a_i = 0 . a_{i-1} a_{i-2} a_{i-3} \ldots$$

of the $a_i$, where we carry digits so that there are infinitely many $j \geq 1$ such that $a_{i-j} \neq 9$. Consider the real number $r(a) \in (0, 1)$ which has decimal expansion

$$r(a) = 0 . r_{-1} r_{-2} r_{-3} \ldots$$

with

$$r_{-2^{m-n}} = a_{m+1,-(n+1)/2}$$
when $m \geq 0$ and when $n \geq 1$ is an odd integer. Is the map $H : W \rightarrow (0,1)$ given by $a \rightarrow r(a)$ injective?

**More Extra Credit Problems**

Note: These problems are optional!

4. A real valued function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous if for every pair of real numbers $x$ and $\epsilon$ with $\epsilon > 0$, there is a real number $\delta$ such that $|f(x) - f(y)| < \epsilon$ if $y$ is a real number and $|x - y| < \delta$. Show that if $f$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, and $f(z) = g(z)$ for all rational numbers $z$, then $f(x) = g(x)$ for all real numbers $x$, so that $f$ and $g$ are the same function. You can use without proof the fact that for every real number $x$ and every real number $\delta > 0$, there is a rational number $z$ such that $|x - z| < \delta$.

**Hint:** Suppose $f(x) \neq g(x)$ for some $x$. Pick $\epsilon = |f(x) - g(x)|/4$. Use the continuity of $f$ and $g$ to show that there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{and} \quad |g(x) - g(y)| < \epsilon$$

if $|x - y| < \delta$. Now pick a rational number $z$ so that $|x - z| < \delta$, and consider the absolute value of

$$f(x) - g(x) = (f(x) - f(z)) + (f(z) - g(z)) + (g(z) - g(x)).$$

You can use $|A + B + C| \leq |A| + |B| + |C|$ for all real numbers $A, B, C$. Do you get a contradictory upper bound on $|f(x) - g(x)|$?

5. Let $J$ be the set of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$. show that $J$ has the same cardinality as $\mathbb{R}$. Is the cardinality of the set $T$ of all functions $h : \mathbb{R} \rightarrow \mathbb{R}$ equal to the cardinality of $\mathbb{R}$?

**Hint:** Use problem # 4 to show that the map which sends $f \in J$ to the restriction of $f$ to the rational numbers $\mathbb{Q}$ defines an injection from $J$ to the set $S$ of all functions from $\mathbb{Q}$ to $\mathbb{R}$. Then use problem number 3. As for $T$, consider the subset of functions $h$ in $T$ which take values in the two element set $\{0,1\}$. 