MATH 203, PROBLEM SET 3

DUE IN JONATHAN KARIV’S MAILBOX IN THE MATH OFFICE, ROOM 4W1 OF DRL LABS, BY 5 P.M.
ON WEDNESDAY, FEB. 10.

1. RELATIONS, PARTIAL ORDERS, TOTAL ORDERS AND WELL-ORDERED SETS.

Suppose \( A = \{a, b\} \) is a set with two elements. Explain your answers to the following
questions.

1. How many relations \( R \) are there between the elements of \( A \) and \( B = A \)? (Recall
that a relation is a subset of \( A \times B \), and that for \( a \in A \) and \( b \in B \) one has \( aRb \)
exactly when \( (a, b) \in R \).

2. How many of the relations \( R \) in problem \#1 are partial orders \( \leq \)?

3. How many of the partial orders \( \leq \) in part (2) are total orders?

4. How many of the total orders in part (3) make \( A \) into a well-ordered set?

2. CARDINALITIES

Show that the following sets have the same cardinalities. You can use the Schroeder-
Bernstein Theorem, which says that if \( A \) and \( B \) are sets and there are injections \( f : A \to B \)
and \( g : B \to A \), then \( \text{Card}(A) = \text{Card}(B) \).

a. The set \( \mathbb{R} \) of all real numbers.

b. The set \( \mathbb{R} - \{0\} \) of all non-zero real numbers. (Hint: To map \( \mathbb{R} \) into \( \mathbb{R} - \{0\} \) note
that \( r + 1 > 0 \) if \( r \geq 0 \).

c. The set \( \mathbb{R}_{>0} \) of all positive real numbers. (Hint: You can use without proof that
\( x = \exp(\ln(x)) \) for all real \( x > 0 \) and that \( y = \ln(\exp(y)) \) for all real \( y \).

d. The set \( I \) of all real numbers \( r \) in the interval \( 0 < r < 1 \). (Hint: Consider the
function \( r \to \frac{1}{r} - 1 \).

e. The set \( I \times I \) of all pairs \( (a, b) \) of positive real numbers \( a \) and \( b \) for which \( 0 < a, b < 1 \).
(Hint: From the decimal expansions of \( a \) and \( b \), try producing a positive real number
\( r \) with \( 0 < r < 1 \) such that the decimal expansion of \( r \) determines the expansions
of both \( a \) and \( b \).)

f. The set \( \mathbb{R} \times \mathbb{R} \) of all points in the \( x \)-\( y \) plane.

g. The set \( 2^{\mathbb{Z}^+} \) of all subsets of the set \( \mathbb{Z}^+ \) of positive integers.

(Hint: You can use without proof the fact that every real number \( r \) in the range
\( 0 < r < 1 \) has a unique binary decimal expansion
\[.a_1a_2a_3\ldots\]
in which each \( a_j \) is 0 or 1 and the expansion does not end with all 1’s. The real number associated to this expansion is

\[ r = \sum_{i=1}^{\infty} a_i 2^{-i}. \]

How would you use this expansion to produce from \( r \) a subset of \( \mathbb{Z}^+ \)? You can use the fact that there is a bijection between \( 2^{\mathbb{Z}^+} \) and the set of all functions \( f : \mathbb{Z}^+ \rightarrow \{0, 1\} \).

How could you use such an \( f \) to produce the binary expansion of an \( r \) in the range \( 0 < r < 1 \) which does not end with all 1’s and which can be used to determine \( f \)?