

OPTIMAL SELECTION BASED ON RELATIVE RANK* (the "Secretary Problem")

BY

Y. S. CHOW, S. MORIGUTI, H. ROBBINS AND S. M. SAMUELS

ABSTRACT

n rankable persons appear sequentially in random order. At the i th stage we observe the relative ranks of the first i persons to appear, and must either select the i th person, in which case the process stops, or pass on to the next stage. For that stopping rule which minimizes the expectation of the absolute rank of the person selected, it is shown that as $n \rightarrow \infty$ this tends to the value

$$\prod_{j=1}^{\infty} \left(\frac{j+2}{j} \right)^{1/j+1} \cong 3.8695.$$

1. Introduction. n girls apply for a certain position. If we could observe them all we could rank them absolutely with no ties, from best (rank 1) to worst (rank n). However, the girls present themselves one by one, in random order, and when the i th girl appears we can observe only her rank relative to her $i-1$ predecessors, that is, $1 +$ the number of her predecessors who are better than she. We may either select the i th girl to appear, in which case the process ends, or reject her and go on to the $(i+1)$ th girl; in the latter case the i th girl cannot be recalled. We must select one of the n girls. Let X denote the absolute rank of the girl selected. The values of X are $1, \dots, n$, with probabilities determined by our selection strategy. What selection strategy (*i.e.* stopping rule) will minimize the expectation $EX =$ expected absolute rank of the girl selected?

To formulate the problem mathematically, let x_1, \dots, x_n denote a random permutation of the integers $1, \dots, n$, all $n!$ permutations being equally likely. The integer 1 corresponds to the best girl, \dots, n to the worst. For any $i = 1, \dots, n$ let $y_i = 1 +$ number of terms x_1, \dots, x_{i-1} which are $< x_i$ ($y_i =$ relative rank of i th girl to appear). It is easy to see that the random variables y_1, \dots, y_n are independent, with the distribution

$$(1) \quad P(y_i = j) = \frac{1}{i} \quad (j = 1, \dots, i),$$

and that

$$(2) \quad P(x_i = k \mid y_1 = j_1, \dots, y_{i-1} = j_{i-1}, y_i = j) = P(x_i = k \mid y_i = j) \\ = \binom{k-1}{j-1} \binom{n-k}{i-j} / \binom{n}{i},$$

Received August 22, 1964

*Research supported by Office of Naval Research and Aerospace Research Laboratories. Reproduction in whole or in part is permitted for any purpose of the United States Government.

so that

$$(3) \quad E(x_i | y_i = j) = \sum_{k=1}^n kP(x_i = k | y_i = j) = \frac{n+1}{i+1} j.$$

For any stopping rule τ the expected absolute rank of the girl selected is therefore $EX = E\left(\frac{n+1}{\tau+1} y_\tau\right)$. We wish to minimize this value by optimal choice of τ .

To find an optimal τ by the usual method of backward induction we define for $i = 0, 1, \dots, n-1$, $c_i = c_i(n)$ = minimal possible expected absolute rank of girl selected if we must confine ourselves to stopping rules τ such that $\tau \geq i+1$. We are trying to find the value c_0 . Now

$$(4) \quad c_{n-1} = E\left(\frac{n+1}{n+1} y_n\right) = \frac{1}{n} \sum_1^n j = \frac{n+1}{2},$$

and for $i = n-1, n-2, \dots, 1$,

$$(5) \quad c_{i-1} = E\left(\min\left(\frac{n+1}{i+1} y_i, c_i\right)\right) = \frac{1}{i} \sum_{j=1}^i \min\left(\frac{n+1}{i+1} j, c_i\right).$$

These equations allow us to compute successively the values $c_{n-1}, c_{n-2}, \dots, c_1, c_0$ and contain the implicit definition of an optimal stopping rule. Equation (5) can be rewritten more simply if we denote by $[x]$ the greatest integer $\leq x$ and set

$$(6) \quad s_i = \left[\frac{i+1}{n+1} c_i \right] \quad (i = n-1, \dots, 1);$$

then (5) becomes

$$(7) \quad \begin{aligned} c_{i-1} &= \frac{1}{i} \left\{ \frac{n+1}{i+1} (1+2+\dots+s_i) + (i-s_i) c_i \right\} \\ &= \frac{1}{i} \left\{ \frac{n+1}{i+1} \cdot \frac{s_i(s_i+1)}{2} + (i-s_i) c_i \right\}. \end{aligned}$$

Defining $s_n = n$, an optimal stopping rule is, stop with the first $i \geq 1$ such that $y_i \leq s_i$; the expected absolute rank of the girl selected using this rule is c_0 .

We observe from (4) and (5) that

$$(8) \quad c_0 \leq c_1 \leq \dots \leq c_{n-1} = \frac{n+1}{2},$$

and from (6) and (8) that $s_i \leq i$ and

$$(9) \quad s_1 \leq s_2 \leq \dots \leq s_{n-1} = \left[\frac{n}{2} \right].$$

For example, let $n = 4$. Then from (4), (6), and (7),

$$c_3 = \frac{5}{2}, s_3 = 2, c_2 = \frac{1}{3} \left(\frac{5}{4} \cdot \frac{2.3}{2} + \frac{5}{2} \right) = \frac{25}{12}, s_2 = 1,$$

$$c_1 = \frac{1}{2} \left(\frac{5}{3} \cdot \frac{1.2}{2} + \frac{25}{12} \right) = \frac{15}{8}, s_1 = 0, c_0 = c_1 = \frac{15}{8},$$

and an optimal stopping rule is given by the vector $(s_1, \dots, s_4) = (0, 1, 2, 4)$. The values of c_0 for $n = 10, 100, 1000$ are found by similar computation to be respectively 2.56, 3.60, 3.83.

D. V. Lindley [1] has treated this problem heuristically for large n by replacing (7) by a single differential equation. His results indicate that for $n \rightarrow \infty$, c_0 should approach a finite limit, but his method is too rough to give the value of this limit. A more adequate but still heuristic approach involves replacing (7) by an infinite sequence of differential equations, one for each value $0, 1, \dots$ of s_i . This method indicates that $\lim c_0$ (all limits as $n \rightarrow \infty$) has the value

$$(10) \quad \prod_{j=1}^{\infty} \left(\frac{j+2}{j} \right)^{1/j+1} \cong 3.8695.$$

It is not clear how to make this heuristic argument rigorous by appealing to known theorems on the approximation of difference equations by differential equations. Instead, we shall give a direct proof that c_0 tends to the value (10).

2. The basic inequalities. We shall derive rather crude upper and lower bounds for the constants c_i which will permit the evaluation of $\lim c_0$. To this end we define the constants

$$(1) \quad t_i = \frac{i+1}{n+1} c_i \quad (i = 0, 1, \dots, n-1).$$

From (1.8) it follows that

$$(2) \quad t_0 < t_1 < \dots < t_{n-1} = \frac{n}{2}$$

and (1.7) becomes

$$(3) \quad t_{i-1} = \frac{s_i(s_i+1) + 2(i-s_i)t_i}{2(i+1)}, \quad s_i = [t_i] \quad (i = 1, \dots, n-1).$$

For fixed i set

$$(4) \quad t_i = s_i + \alpha, \quad 0 \leq \alpha < 1;$$

then (3) becomes

$$(5) \quad t_{i-1} = \frac{t_i(1+2i-t_i)}{2(i+1)} - \frac{\alpha(1-\alpha)}{2(i+1)}.$$

Put

$$(6) \quad T(x) = \frac{x(1+2i-x)}{2(i+1)};$$

then for $x \leq i + \frac{1}{2}$,

$$(7) \quad T'(x) = \frac{1+2i-2x}{2(i+1)} \geq 0$$

so $T(x)$ is increasing for $x \leq i + \frac{1}{2}$, and by (5),

$$(8) \quad t_{i-1} \leq T(t_i).$$

We now prove the first basic inequality.

LEMMA 1.

$$(9) \quad t_i \leq \frac{2n}{n-i+3} \quad (i=0, \dots, n-1).$$

Proof. (9) is true for $i = n-1$ since by (2)

$$t_{n-1} = \frac{n}{2} = \frac{2n}{4}.$$

Assume (9) holds for some $1 \leq i \leq n-1$; we shall prove that it holds for $i-1$. We know by (1.8) and (1) that

$$(10) \quad t_{i-1} = \frac{i}{n+1} c_{i-1} \leq \frac{i}{n+1} \cdot \frac{n+1}{2} = \frac{i}{2}.$$

If $\frac{i}{2} \leq \frac{2n}{n-i+4}$ then (9) holds for $i-1$. If $\frac{i}{2} > \frac{2n}{n-i+4}$ then $i + \frac{1}{2} \geq \frac{2n}{n-i+3}$, so by (8), since $T(x)$ is increasing for $x \leq i + \frac{1}{2}$,

$$(11) \quad t_{i-1} \leq T(t_i) \leq T\left(\frac{2n}{n-i+3}\right) = \frac{n\{(1+2i)(n-i+3)-2n\}}{(i+1)(n-i+3)^2} \leq \frac{2n}{n-i+4},$$

since the last inequality is equivalent to

$$(n-i+3)^2 + (2n-2i-1)(n-i+3) + 2n \geq 0,$$

which is true for $n-i \geq 1$. Hence (9) holds for $i-1$ in this case also, and the lemma is proved.

COROLLARY 1.

$$(12) \quad c_0 < 8 \quad (n=1, 2, \dots).$$

Proof. In (9) set $i = \left\lfloor \frac{n}{2} \right\rfloor$. Then

$$(13) \quad c_0 \leq c_i = \frac{n+1}{i+1} t_i \leq \frac{n+1}{i+1} \cdot \frac{2n}{n-i+3} \leq \frac{2n(n+1)}{\frac{n}{2} \left(\frac{n}{2} + 3 \right)} < 8.$$

We next observe that

$$(14) \quad t_{i-1} \geq \frac{i}{i+1} \left\{ 1 - \frac{t_i}{2(i+1)} \right\} \quad (i = 1, \dots, n-1).$$

This inequality follows from (5) if we show that

$$\frac{t_i(1+2i-t_i)}{2(i+1)} - \frac{\alpha(1-\alpha)}{2(i+1)} \geq \frac{i}{i+1} t_i \left\{ 1 - \frac{t_i}{2(i+1)} \right\},$$

which reduces to

$$(15) \quad t_i \left(1 - \frac{t_i}{i+1} \right) \geq \alpha(1-\alpha) \quad (\alpha = t_i - [t_i]),$$

and this is true if $t_i \geq 1$, since $t_i \leq \frac{i+1}{2}$ by (10), and if $t_i < 1$, since then $t_i = \alpha$.

We now establish the second basic inequality.

LEMMA 2.

$$(16) \quad t_i \geq \frac{3(i+1)}{2(n-i+2)} \quad (i = 0, \dots, n-1).$$

Proof. (16) is true for $i = n-1$; suppose it is true for some $1 \leq i \leq n-1$. Define

$$T(x) = x \left\{ 1 - \frac{x}{2(i+1)} \right\},$$

which is increasing for $x \leq i+1$. Since by (10) $t_i \leq \frac{i+1}{2}$, we have by (14),

$$\begin{aligned} t_{i-1} &\geq \frac{i}{i+1} T(t_i) \geq \frac{i}{i+1} T\left(\frac{3(i+1)}{2(n-i+2)}\right) \\ &= \frac{3i(4n-4i+5)}{8(n-i+2)^2} \geq \frac{3i}{2(n-i+3)}. \end{aligned}$$

which is equivalent to $i \leq n-1$. This proves the lemma.

We have seen (1.9) that for any positive integer k , if $n \geq 2k$ then $s_{n-1} \geq k$. We now define for each $k = 1, 2, \dots$ and each $n \geq 2k$,

$$(17) \quad i_k = \text{smallest integer } j \geq 1 \text{ such that } s_j \geq k.$$

We note that $s_{i_1-1} = 0$ and hence from (1.7),

$$(18) \quad c_0 = c_1 = \dots = c_{i_1-1}.$$

COROLLARY 2.

$$(19) \quad \lim_{\underline{\quad}} \frac{i_1}{n} \geq \frac{1}{8}.$$

Proof. If $i_1 > [n/2]$ then $i_1 \geq [n/2] + 1 > n/2$, $(i_1/n) > \frac{1}{2}$. If $i_1 \leq [n/2]$ then by (13),

$$1 \leq s_{i_1} \leq t_{i_1} = \frac{i_1 + 1}{n + 1} c_{i_1} \leq \frac{i_1 + 1}{n + 1} c_{[n/2]} < \frac{i_1 + 1}{n + 1} \cdot 8.$$

(We remark that $i_1 > 1$ for $n > 2$, since if $i_1 = 1$ then $s_1 = 1$ and $c_0 = \frac{n + 1}{2}$, which only holds for $n \leq 2$.)

COROLLARY 3. *On every set*

$$(20) \quad \left\{ \alpha \leq \frac{i}{n} \leq \beta; 0 < \alpha < \beta < 1 \right\},$$

$$(21) \quad \lim(t_i - t_{-1}) = 0$$

uniformly.

Proof. From (14) and (9),

$$\begin{aligned} 0 \leq t_i - t_{i-1} &\leq t_i - \frac{i}{i+1} t_i \left\{ 1 - \frac{t_i}{2(i+1)} \right\} = \frac{t_i}{i+1} + \frac{it_i^2}{2(i+1)^2} \\ &\leq \frac{(1+t_i)^2}{2(i+1)} \leq \frac{\left(1 + \frac{2n}{n-i}\right)^2}{2(i+1)} \leq \frac{\left(1 + \frac{2}{1-\beta}\right)^2}{2\alpha} \cdot \frac{1}{n} \rightarrow 0. \end{aligned}$$

COROLLARY 4. *For $k = 1, 2, \dots$ and $n \geq 12k$,*

$$(22) \quad \frac{i_k}{n} \geq 1 - \frac{2}{k},$$

$$(23) \quad \frac{i_k}{n} \leq 1 - \frac{1}{2k}.$$

Proof. By (9),

$$s_{i_k} \geq k \Rightarrow t_{i_k} \geq k \Rightarrow \frac{2n}{n - i_k} \geq k \Rightarrow \frac{i_k}{n} \geq 1 - \frac{2}{k},$$

which proves (22). (23) holds if $i_k \leq \left[\frac{n}{2} \right]$, for then

$$\frac{i_k}{n} \leq \frac{1}{2} \leq 1 - \frac{1}{2k},$$

and if $i_k > \left\lceil \frac{n}{2} \right\rceil$ then by (16),

$$s_{i_k-1} < k \Rightarrow t_{i_k-1} < k \Rightarrow \frac{3i_k}{2(n-i_k+3)} < k \Rightarrow \frac{3i_k}{2k} < n - i_k + 3 \Rightarrow \frac{3n}{4k} < n - i_k + 3$$

$$\Rightarrow \frac{3}{4k} < 1 - \frac{i_k}{n} + \frac{3}{n} \Rightarrow \frac{i_k}{n} < 1 - \frac{3}{4k} + \frac{3}{n} \leq 1 - \frac{1}{2k} \text{ for } n \geq 12k.$$

COROLLARY 5.

$$(24) \quad \lim t_{i_k} = \lim t_{i_k-\gamma} = k \quad (k, \gamma = 1, 2, \dots).$$

Proof. $t_{i_k-\gamma} < k \leq t_{i_k}$.

Choose α, β so that $0 < \alpha < \frac{1}{8}, 1 - \frac{1}{2k} < \beta < 1$. Then by (19) and (23),

$$(25) \quad \alpha < \frac{i_k - \gamma}{n} < \frac{i_k}{n} < \beta$$

for sufficiently large n . Hence by Corollary 3,

$$\lim(t_{i_k} - t_{i_k-\gamma}) = 0.$$

COROLLARY 6. For $k = 1, 2, \dots$

$$(26) \quad s_{i_k} = k \text{ for sufficiently large } n,$$

$$(27) \quad \lim(i_{k+1} - i_k) = \infty.$$

Proof. $k \leq s_{i_k} \leq t_{i_k}$ together with (24) proves (26).

$$\lim(t_{i_{k+1}} - t_{i_k}) = 1 \text{ and } \lim(t_{i_{k+1}} - t_{i_{k+1}-\gamma}) = 0$$

by (24), and these relations imply (27).

3. **Proof of the Theorem.** Choose and fix a positive integer k and let n by (2.26) be so large that $s_{i_k} = k, s_{i_{k+1}} = k + 1$. For $i_k \leq i < i_{k+1}$ define

$$(1) \quad v_i = t_i - \frac{k}{2}.$$

Substituting in (2.3) we find that

$$v_{i-1} + \frac{k}{2} = \frac{k(k+1) + 2(i-k) \left(v_i + \frac{k}{2} \right)}{2(i+1)} = \frac{k}{2} + \frac{i-k}{i+1} v_i,$$

$$(2) \quad v_i = \frac{i+1}{i-k} v_{i-1}.$$

Hence for $i_k < i < i_{k+1}$,

$$(3) \quad v_i = \frac{i+1}{i-k} v_{i-1} = \frac{i+1}{i-k} \frac{i}{i-k-1} \cdots \frac{i_k+2}{i_k-k+1} v_{i_k}$$

$$= \frac{i+1}{i_k+1} \cdot \frac{i}{i_k} \cdots \frac{i+1-k}{i_k+1-k} \cdot v_{i_k} = v_{i_k} \prod_{j=1}^{k+1} \left(\frac{i+j-k}{i_k+j-k} \right),$$

and hence

$$(4) \quad t_i = \frac{k}{2} + \left(t_{i_k} - \frac{k}{2} \right) \prod_{j=1}^{k+1} \left(\frac{i+j-k}{i_k+j-k} \right).$$

Set $i = i_{k+1} - 1$; then

$$(5) \quad t_{i_{k+1}-1} = \frac{k}{2} + \left(t_{i_k} - \frac{k}{2} \right) \prod_{j=1}^{k+1} \left(\frac{i_{k+1}+j-k-1}{i_k+j-k} \right).$$

From (2.19) and (2.24) it follows that

$$k+1 = \frac{k}{2} + \frac{k}{2} \prod_{j=1}^{k+1} \lim \left(\frac{i_{k+1}+j-k-1}{i_k+j-k} \right) = \frac{k}{2} + \frac{k}{2} \lim \left(\frac{i_{k+1}}{i_k} \right)^{k+1}$$

and hence

$$(6) \quad \lim \frac{i_{k+1}}{i_k} = \left(\frac{k+2}{k} \right)^{1/k+1}, \quad \lim \frac{i_1}{i_k} = \lim \frac{i_1/n}{i_k/n} = \prod_{j=1}^{k-1} \left(\frac{j}{j+2} \right)^{1/j+1}.$$

From (2.22)

$$(7) \quad \left(1 - \frac{2}{k} \right) \prod_{j=1}^{k-1} \left(\frac{j}{j+2} \right)^{1/j+1} \leq \underline{\lim} \frac{i_1}{n} \leq \overline{\lim} \frac{i_1}{n} \leq \prod_{j=1}^{k-1} \left(\frac{j}{j+2} \right)^{1/j+1}.$$

Letting $k \rightarrow \infty$,

$$(8) \quad \lim \frac{i_1}{n} = \prod_{j=1}^{\infty} \left(\frac{j}{j+2} \right)^{1/j+1}.$$

Now by (2.24) and (2.18),

$$(9) \quad 1 = \lim t_{i-1} = \lim \left(\frac{i_1}{n+1} c_{i-1} \right) = \lim \left(\frac{i_1}{n} c_0 \right)$$

$$= \lim c_0 \cdot \prod_{j=1}^{\infty} \left(\frac{j}{j+2} \right)^{1/j+1}.$$

Thus we have proved the

THEOREM.
$$\lim c_0 = \prod_{j=1}^{\infty} \left(\frac{j+2}{j} \right)^{1/j+1}.$$

4. Remarks.

1. It is interesting to note that $c_0 = c_0(n)$ is a strictly increasing function of n ; thus in view of the Theorem,

$$(1) \quad c_0(n) < 3.87 \quad (n = 1, 2, \dots).$$

A direct proof that $c_0(n)$ is strictly increasing based on the formulas of Section 1 is difficult, since there is no obvious relation between the c_i for different values of n . However, a direct probabilistic proof can be given which involves no use of the recursion formulas. Let $(s_1, \dots, s_n, n+1)$ be any stopping rule for the case of $n+1$ girls such that $s_i < i$ for $i = 1, \dots, n$ (any optimal stopping rule has this property). Define for $i = 1, \dots, n$

$$(2) \quad t_j(i) = \begin{cases} s_j & \text{for } j = 1, \dots, i-1, \\ s_{j+1} & \text{for } j = i, \dots, n, \end{cases}$$

and

$$(3) \quad t_j(n+1) = \begin{cases} s_j & \text{for } j = 1, \dots, n-1, \\ n & \text{for } j = n. \end{cases}$$

It is easy to see that at least one of the stopping rules defined by $(t_1(i), \dots, t_n(i))$, $i = 1, \dots, n+1$ must give a value of c_0 for the case n which is less than that given by $(s_1, \dots, s_n, n+1)$ for the case $n+1$. Hence $c_0(n) < c_0(n+1)$.

2. We assumed that the n girls appear in random order, all $n!$ permutations being equally likely. The minimal expected absolute rank of the girl chosen is then $c_0 < 3.87$ for all n . Suppose now that the order in which the girls are to appear is determined by an opponent who wishes to maximize the expected absolute rank of the girl we choose. No matter what he does, by choosing at random the first, second, \dots , last girl to appear we can achieve the value

$$EX = (1 + 2 + \dots + n)/n = (n+1)/2.$$

And in fact *there exists an opponent strategy such that, no matter what stopping rule we use, $EX = (n+1)/2$* . Let $x_1 = 1$ or n , each with probability $1/2$; let $x_{i+1} =$ either the largest or the smallest of the integers remaining after x_1, \dots, x_i have been chosen, each with probability $1/2$. If we define for $i = 1, \dots, n$

$$(4) \quad z_i = E(x_i | y_1, \dots, y_i), \quad \mathcal{B}_i = \mathcal{B}(y_1, \dots, y_i),$$

then it is easy to see that $\{z_i, \mathcal{B}_i (i = 1, \dots, n)\}$ is a martingale, so that for any stopping rule τ ,

$$(5) \quad E(X) = E(z_\tau) = E(z_1) = (n+1)/2.$$

Many extensions and generalizations of the problem considered in this paper suggest themselves at once. Some further results will be presented elsewhere.

REFERENCE

1. D. V. Lindley, *Dynamic programming and decision theory*, Applied Statistics **10** (1961), 39-51.

PURDUE UNIVERSITY
UNIVERSITY OF TOKYO
COLUMBIA UNIVERSITY
PURDUE UNIVERSITY