1. Independent random variables

Let \( X : S \to \mathbb{R} \) be a random variable. The distribution function of \( X \) is the function 
\[ F_X : \mathbb{R} \to \mathbb{R} \]
defined by
\[ F_X(t) = \text{Prob}(X \leq t). \]
Random variables \( X_1, \ldots, X_n \) are independent if for all \( n \)-tuples of closed intervals \([a_1, b_1], \ldots, [a_n, b_n] \), one has
\[ \text{Prob}(X_i \text{ is in } [a_i, b_i] \text{ for all } i = 1, \ldots, n) = \prod_{i=1}^{n} \text{Prob}(X_i \in [a_i, b_i]). \]

1. Suppose that at time \( t = 0 \) you begin watching a TV program in which a liberal, a moderate and a conservative are discussing politics. Let \( X_1 \) be the amount of time before the liberal says the word “extremists.” Let \( X_2 \) be the amount of time before the moderate stands up and walks out. Let \( X_3 \) be the amount of time before the conservative says “founding fathers.” You decide you are going to switch off the program as soon as either:
   (i) The moderate walks out, or
   (ii) The liberal has said “extremist” and the conservative has said “founding fathers”.

Let \( X \) be the random variable giving the time you switch off the program. Suppose that \( X_1, X_2 \) and \( X_3 \) are independent. How can you write down the distribution function \( F_X \) of \( X \) in terms of the distribution functions \( F_{X_1}, F_{X_2} \) and \( F_{X_3} \)?

(Hint: To find \( F_X(t) = \text{Prob}(X \leq t) \), try to write down the event that \( X \leq t \) in terms of conditions satisfied by \( X_1, X_2 \) and \( X_3 \). Then try to write this event as a disjoint union of events to which you can apply the independence of \( X_1, X_2 \) and \( X_3 \).)

2. Write down some examples form your own experience of independent random variables, and of random variables which are not independent. As challenge, try to think of a real life situation in which there are three random variables which are pairwise independent but which are not independent.

2. Expectations

Suppose that a movie producer is considering whether to show their new film, “Saw LXIII: Stubbed toes and Paper cuts,” in theaters. The producer tries to determine whether this will be profitable for them in the following way.

Suppose \( N \) is the total number of people who will see the movie at some point. Let \( X \) be the random variable given by the average time in weeks that someone sees the movie,
assuming that they see it at some point. Assume that the probability density function of
$X$ has the form
$$f_X(t) = c_1 e^{c_2 t}$$
for some constants $c_1$ and $c_2$. Thus the probability that someone who will eventually see
it will go to see it between times $a$ and $b$ is
$$\text{Prob}(a \leq X \leq b) = \int_a^b f_X(t) \, dt,$$
when $0 \leq a \leq b$.

3. Suppose we measure time in units of weeks, and that the odds that someone sees
the move in week two (from time 1 to time 2) are half what they are in week one
(from time 0 to time 1) if they see the movie at all. What should $c_1$ and $c_2$ be in
the formula for $f_X(t)$?

4. Using $c_1$, $c_2$ and $N$ write down a formula for the expected number $h(T)$ of people
who will have seen the move by time $T$.

5. Suppose that the cost of showing the movie for $T$ weeks is given by $g(T) = I + cT$
dollars, where $I$ is the start up cost and $c$ is a constant. Suppose each person who
sees the move brings in $r$ dollars of revenue. How would you use the functions $h(T)$,
g($T$) and $r$ to determine whether there is a number $T$ of weeks to show the movie
which will be profitable for the movie maker? (Try to push the calculus as far as
you can to arrive at a readily testable formula.) Is it ever profitable for the theater
to show the movie forever?

3. The semester project

6. Describe an idea you’d like to pursue for your semester project. Anything of interest
to you which has some mathematical content is worth considering, and I’ll be happy
to talk with you and your group about this. Some ideas which have come up during
this semester are:
   i. Follow through on optimizing the expected rank of the person selected by the
      interview procedure we talked about in class.
   ii. Look into the statement made on page 81 of the book “How math can save
      your life” about the effect on life expectancy of freeway driving.
   iii. How can probability and game theory be used to come up with a system for
       settling legal cases?
But please don’t feel constrained to follow these ideas!