1. The central limit theorem

In class we talked about the central limit theorem for \( n \geq 1 \) independent random variables \( X_1, \ldots, X_n \) having the same distribution function. Let \( \mu = E(X_i) \) and \( \sigma = \sigma(X_i) = \sqrt{\text{Var}(X_i)} \) be the mean and standard deviation of each of the \( X_i \). Define

\[
Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}}.
\]

The central limit theorem, with Berry’s error term, says

\[
|\Pr(Y_n \leq x) - \int_{-\infty}^{x} \phi(u) du| \leq \frac{3\rho}{\sigma^3 \sqrt{n}}
\]

when \( \rho = E(|X_i - \mu|^3) \), where \( \phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \) is the probability density function of the normal distribution.

1. The newspaper columnist Marilyn Vos Savant often claimed that she had an I. Q. of 220, which would be 8 standard deviations above the mean. Suppose that the I.Q. test supporting this had \( n \) questions. Suppose that scores of people taking the test on questions 1 through \( n \) are represented by independent random variables \( X_1, \ldots, X_n \), with Prob\((X_i = 1) = 1/2\) being the probability of a correct answer and Prob\((X_i = 0) = 1/2\) being the probability of a wrong answer. What is the smallest number \( n \) of questions on the exam such that Marilyn Vos Savant’s total score \( X = X_1 + \cdots + X_n \) could have been 8 standard deviations above the mean? (Hint: You don’t need to use the central limit theorem for this, just the formula for \( \sigma(X) \).)

2. When \( n \) is your answer to question 1, what is the probability that a person has the same score on the exam as Marilyn Vos Savant? If the population of the earth is \( 6 \cdot 10^9 \), what is the probability of finding a person with this high a score on earth, given that the exam could be modeled as in problem #1? How would you explain your conclusions?

3. Suppose we now want to use the central limit theorem to approximate the probability found in question #2. Let \( x = 8 \) in (1.1). How large should we make \( n \) in order for to make the error term on the right hand side of (1.1) less than the probability you found in question #2 of a person having a score equal to Marilyn Vos Savant’s? (This \( n \) will be considerably larger than the answer you find in question #1.)

Comment: This is another illustration of how the known error estimates for the central limit theorem do not work well at large distances from the mean unless \( n \) is exceedingly large.
2. Stirling’s formula

One form of Stirling’s formula is that if \( n \geq 1 \) is an integer, then

\[
(2.2) \quad n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\lambda_n}
\]

where

\[
(2.3) \quad \frac{1}{12n+1} < \lambda_n < \frac{1}{12n}
\]

4. Use Stirling’s formula to show that the probability of getting exactly \( n/2 \) heads on flipping a fair coin an even number \( n \) of times is equal to

\[
(2.4) \quad \sqrt{\frac{2}{n\pi}} e^{s_n}
\]

where

\[
\frac{1}{12n+1} - \frac{1}{3n} < s_n < \frac{1}{12n} - \frac{2}{6n+1}.
\]

(This corrects a formula we talked about in class.)