1. **Constant coefficient systems of linear O.D.E.’s**.

The object of the problems on the next page is to complete the proof of a result about matrix exponentials.

Suppose $A = (a_{i,j})_{1 \leq i,j \leq n}$ is an $n \times n$ matrix of complex numbers. Let $t$ be a real variable. We showed in class that the matrix exponential function

\begin{equation}
\label{eq:1.1}
e^{At} = \sum_{m=0}^{\infty} \frac{A^m t^m}{m!}
\end{equation}

is well defined. Further, the series expression for each matrix entry of $e^{At}$ which results from the right hand side of (1.1) is absolutely convergent for all $t$. In this case, one says the matrix series converges absolutely.

We began the proof that

\begin{equation}
\label{eq:1.2}
\frac{d}{dt}(e^{At}) = Ae^{At}
\end{equation}

in the following way.

By definition,

\[
\frac{d}{dt}(e^{At}) = \lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h}
\]

Since the series (1.1) is absolutely convergent, we can rearrange terms to have

\[
\frac{e^{A(t+h)} - e^{At}}{h} = \sum_{m=0}^{\infty} A^m \left( \frac{(t+h)^m - t^m}{m!h} \right)
\]

\[
= \sum_{m=1}^{\infty} A^m \left( \frac{\left(\sum_{\ell=0}^{m} \binom{m}{\ell} t^{m-\ell} h^\ell\right) - t^m}{m!h} \right)
\]

\[
= \sum_{m=1}^{\infty} A^m \frac{t^{m-1}}{(m-1)!} + \sum_{m=1}^{\infty} A^m \left( \frac{\sum_{\ell=2}^{m} \binom{m}{\ell} t^{m-\ell} h^{\ell-1}}{m!} \right)
\]

\begin{equation}
\label{eq:1.3}
= Ae^{At} + \sum_{\ell=2}^{\infty} \sum_{m=\ell}^{\infty} A^m \left( \frac{\binom{m}{\ell} t^{m-\ell} h^{\ell-1}}{m!} \right)
\end{equation}

Here the second line on the right follows by the binomial theorem, and the $m = 0$ term is 0. The third line follows from the second on considering the $\ell = 0$, the $\ell = 1$ and the $\ell > 1$ terms. The fourth line is a rearrangement of the third, using the definition of $e^{At}$. 

Problem 1. Deduce from (1.3) that
\[ e^{A(t+h)} - e^{At} = A e^{At} + \sum_{\ell=2}^{\infty} \sum_{m=\ell}^{\infty} A^m \left( \frac{t^{m-\ell} h^{\ell-1}}{(m-\ell)! \ell!} \right) \]

Problem 2. View \( t \) as fixed. Suppose one can show that for all real \( h \) in the range \(|h| \leq 1\) the following is true. The series
\[ T(t, h) = \sum_{\ell=2}^{\infty} \sum_{m=\ell}^{\infty} A^m \left( \frac{t^{m-\ell} h^{\ell-2}}{(m-\ell)! \ell!} \right) \]
is absolutely convergent for all \( h \) in the range \(|h| \leq 1\), and that \(|T(t, h)|\) is bounded for \( h \) in this range when \( t \) is fixed. (Recall that \(|T(t, h)|\) is the maximum of the absolute values of the entries of \( T(t, h) \).) Explain why this implies that the second term on the right side of (1.4) is
\[ \sum_{\ell=2}^{\infty} \sum_{m=\ell}^{\infty} A^m \left( \frac{t^{m-\ell} h^{\ell-1}}{(m-\ell)! \ell!} \right) = hT(t, h) \]

Then show that
\[ \lim_{h \to 0} hT(t, h) \]
is the zero matrix, and that (1.2) is true. Why do you need some control on \(|T(t, h)|\) as \( h \to 0\)?

Problem 3. In class we showed that \(|A^\ell| \leq |A|^{\ell n^{\ell-1}}\) for \( \ell \geq 1\), where \(|A|\) is the maximum of the absolute values of the entries of \( A \). Use this to show that the matrix series
\[ B(h) = \sum_{\ell=2}^{\infty} A^\ell \frac{h^{\ell-2}}{\ell!} \]
is absolutely convergent, with
\[ |B(h)| \leq \sum_{\ell=2}^{\infty} |A|^\ell n^{\ell-1} \frac{h^{\ell-2}}{\ell!} = e^{|A|nh} - 1 - |A|nh \]
for \( h \neq 0\). Use the power series for \( e^{|A|nh} \) to deduce that there is a constant \( M' \) depending on \( n \) and \(|A|\) so that \(|B(h)| \leq M' \) for \(|h| \leq 1\). (Hint: A convergent series of positive increasing functions is an increasing function.)

Problem 4. Use problem 3 to show that the matrix series \( T(t, h) \) in Problem 2 is absolutely convergent and bounded by the following product of absolutely convergent matrix series:
\[ \left( \sum_{m'=0}^{\infty} \frac{A^m t^{m'}}{m'!} \right) \cdot \left( \sum_{\ell=2}^{\infty} A^\ell \frac{h^{\ell-2}}{\ell!} \right) = e^{At} \cdot B(h) \]

Then explain why and problem 3 show \(|T(t, h)|\) is bounded for \(|h| \leq 1\) and \( t \) fixed. This will complete the proof of (1.2). (Hint: Write \( m \geq \ell \) as \( m = m' + \ell \) where \( m' = m - \ell \geq 0 \)).

Remark One might try to simplify the above proof of (1.2) by showing first that \( e^{A(t+h)} = e^{At} \cdot e^{Ah} \).
2. A POLITICAL STABILITY MODEL

Suppose that \( L = L(t), \) \( C = C(t) \) and \( U = U(t) \) represent the number of liberals, conservatives and uncommitted voters at time \( t \). We will suppose that interactions between liberals and conservatives lead to each become uncommitted with certain probabilities. Uncommitted voters spontaneously become liberal or conservative at a certain rates.

**Problem 5.** Suppose that there are positive constants \( \alpha, \beta, \tau \) and \( \gamma \) such that \( L, C \) and \( U \) satisfy the following differential equations:

\[
\frac{dL}{dt} = -\alpha LC + \tau U \\
\frac{dC}{dt} = -\beta LC + \gamma U \\
\frac{dU}{dt} = (\alpha + \beta) LC - (\tau + \gamma)U
\]

Explain why these equations correspond to the above verbal description of the evolution of \( L, C \) and \( U \).

**Problem 6.** Viewing this as an autonomous system of O.D.E.’s in the vector variable \( x = (L, C, U) \). Suppose that \( \alpha \gamma \neq \tau \beta \). Find all values of \( x \) which are equilibria for this system.

**Problem 7.** Is the system ever linearly stable at the equilibrium points you found in Problem 6?

**Problem 8.** Show that \( L(t) + C(t) + U(t) \) equals some constant \( \kappa \) independent of \( t \). Suppose \( \kappa > 0 \) and \( \alpha \gamma \neq \tau \beta \). Rewrite the system of differential equations as a system just involving \( L \) and \( C \), using that \( U = \kappa - L - C \). When you do this, which of the equilibria you found in Problem 6 are stable for the two variable system involving only \( L \) and \( C \)? Your answer should depend on the \( \alpha, \beta, \tau \) and \( \gamma \). Can you explain why this answer makes heuristic sense?

**Remark** Because there is a conserved quantity \( L + C + U = \kappa \) which does not change with time, stability of the three variable system is not a natural condition. This is because such stability requires that all small variations of \( (L, C, U) \) from an equilibrium return toward the equilibrium, and most of these variations will not conserve \( L + C + U \).

**Extra Credit** What happens in Problems 6 and 8 if \( \alpha \gamma = \tau \beta \)? You can turn this in at any time during the semester.