MATH 210, PROBLEM SET 5

DUE IN LECTURE ON FRIDAY, APRIL 7

1. Partial conflict games

1.1. **Background.** The video posted on March 3 has to do with partial conflict games in which two players can either compete or cooperate with each other. They each rank their preferences for the outcome of their choices by the numbers $1, 2, 3, 4$, with 4 being the **best** outcome for them and 1 being the **worst**. Their choices are thus described by a matrix

	Player II competes	Player II cooperates
Player I competes	(a_1, b_1)	(a_2, b_2)
Player I cooperates	(a_3, b_3)	(a_4, b_4)

in which (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) are permutations of $(1, 2, 3, 4)$.

The two most important concepts are:

Definition 1.1. An option for one player is **dominant** if it is always preferrable for them to their other option, regardless of what the other player does.

Definition 1.2. Suppose player 1 chooses option $s(1)$ and player 2 chooses option $s(2)$, where $s(1)$ and $s(2)$ are in the set $\{$"cooperate", "compete"$\}$. The pair of choices $(s(1), s(2))$ is a **Nash equilibrium** if neither player has a reason to switch their choice if they assume that the other player will continue to make the choice they have made.

Nash equilbria are important because these represent pairs of choices which are stable when the players assume the other player won't switch what they have chosen.

1.2. **Examples.**

1. **The prisoner's dilemma.**

   Imagine two prisoners have been arrested on suspicion of committing a crime together. They are brought to separate rooms for interrogation. Each must decide whether to implicate the other prisoner or not. If they implicate the other prisoner, this amounts to competing with them, since doing this may lead to shifting the blame from themselves. If they don't implicate the other prisoner, this amounts to cooperating with the other prisoner. The prisoner's dilemma payoff matrix is this:

	Player II competes:	Player II cooperates
Player I competes:	$(2,2)$	$(4,1)$
Player I cooperates	$(1,4)$	$(3,3)$

Here is the rationale for these payoffs.

A. The worst outcome for a player is that they are implicated by the other player at the same time they don't implicate the other player. Then the crime is confirmed, and the player receives all the blame; the other player goes free. This explains the $(1,4)$ and $(4,1)$ entries in the matrix.

B. If both players implicate each other, they will be worse off than if they both refuse to implicate the other. That explains the $(2,2)$ and $(3,3)$ entries.

The reason this game is called the prisoner's dilemma is that competing is a dominant option for each player, but this will leave them both worse off than if they both cooperated.

The choice of both competing is a Nash equilibrium. This is because if both players compete, and they assume the other player wil continue to do so, then each player would be worse off switching to cooperating.

In fact, the only Nash equilibrium is when both players compete. For example, player I competing and player II cooperating (which gives preferences $(4,1)$) is not Nash, since if player II assumes player I will compete, they would be better off competing. If both cooperate with the other, this is not Nash, since if either player assumes the other will continue to coopoerate, they would be better off if they switch to competing.

2. **The game of chicken.**

Two people drive cars directly at each other. Each can decide to swerve at the last minute (cooperating with the other player) or they can refuse to swerve (competing). The chicken payoff matrix is this:

	Player II competes:	Player II cooperates
Player I competes:	$(1,1)$	$(4,2)$
Player I cooperates	$(2,4)$	$(3,3)$

Here is the rationale for these payoffs.

A. The worst outcome for both players is that neither swerves, i.e. if they both compete. Then they collide and both go up in a fireball.
B. The best outcome for a player is if they compete (they don't swerve) but the other player cooperates (swerves). Then the competing player looks good, and no one goes up in a fireball. The player who swerves looks like a chicken, but at least they survive.
C. If both players swerve, they are each better off than if they either look like chickens or go up in a fireball. They are not as well off as if they win the war of nerves, though.

There are no dominant options for either player.

The Nash equilibria are when one player swerves and the other doesn't. For example, suppose player I swerves but player II doesn't. If player I assumes player II will not swerve, they are not better off switching, since that would lead to going up in a fireball. Similarly, if player II assumes player I will continue to swerve, they are not better off swerving, since then they would no longer be seen as the winner in the war of nerves. The option of neither swerving is not Nash, since they would both be better off switching. The option of both swerving is also not Nash, since if either assumes the other will not swerve, they would be better off not swerving.

1.3. **Political implications.** Suppose the two players are two competing political parties. The difference between the prisoner's dilemma scenario and the chicken scenario has to do with how disastrous it is for the parties to not cooperate with one another.

Chicken is consistent with not working together being the worst outcome for both. This could be the case, for example, if the voters would punish both parties severely for not cooperating. The fact that the only Nash equilibria involve one party cooperating and the other competing suggests that in this situation, one party will back down and the other will be seen as winning the political confrontation. The more likely party to back down would be the one that has more to lose; this tends to be the party in power. In 2017, for example, the issue will be whether the Republican party will soften its health care bill after the congressional budget office released a report predicting 24 million people would lose their health insurance over 10 years as a result of the original form of the bill.

In the Prisoner's dilemma, it is worse for a party to be seen as losing a confrontation than it is to be seen as intransigent and getting nothing done. The only Nash equilbrium is when neither cooperates.

For example, during a political campaign, the two parties may be behaving more along the lines of the Prisoner's dilemma. After an election, they may find themselves playing a game of chicken when it comes to legislation which will affect their constituents and their political futures.

1.4. **Homework Problems about partial conflict games.** Suppose that as before, a partial conflict game has matrix

	Player II competes:	Player II cooperates
Player I competes:	(a_1, b_1)	(a_2, b_2)
Player I cooperates	(a_3, b_3)	(a_4, b_4)

Here each of $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ are permutations of $\{1, 2, 3, 4\}$. Suppose that the entries satisfy these conditions:

A. $a_1 = b_1$, $a_4 = b_4$, $a_2 = b_3$ and $a_3 = b_2$.
B. $a_3 < a_4$ and $a_1 < a_2$.

Problem 1. Explain why condition (A) is equivalent to the players having the same preferences regarding competing or cooperating against the other player.

Problem 2. Explain in words what condition (B) says about the preferences of a player regarding the behavior of the other player. Is this a reasonable assumption or not?

Problem 3. How many different preference matrices are there which satisfy both (A) and (B)? Indicate which of these are Chicken and which are the Prisoner's Dilemma.

Problem 4. For all of the preference matrices in # 3 which are neither Chicken nor the Prisoner's Dilemma, find all of the Nash equilibria. Then make up a real world scenario which might lead to this set of preferences.

2. A political stability model

Suppose that $L = L(t)$, $C = C(t)$ and $U = U(t)$ represent the number of liberals, conservatives and uncommitted voters at time t. We will suppose that interactions between liberals and conservatives lead to each become uncommitted with certain probabilities. Uncommitted voters spontaneously become liberal or conservative at a certain rates.

Problem 5. Suppose that there are positive constants α, β, τ and γ such that L, C and U satisfy the following differential equations:

$$\frac{dL}{dt} = -\alpha LC + \tau U \tag{2.1}$$

$$\frac{dC}{dt} = -\beta LC + \gamma U \tag{2.2}$$

$$\frac{dU}{dt} = (\alpha + \beta)LC - (\tau + \gamma)U \tag{2.3}$$

Explain why these equations correspond to the above verbal description of the evolution of L, C and U.

Problem 6. Viewing this as an autonomous system of O.D.E.'s in the vector variable

$$x(t) = \begin{pmatrix} \mathrm{L(t)} \\ \mathrm{C(t)} \\ \mathrm{U(t)} \end{pmatrix}.$$

Suppose that $\alpha\gamma \neq \tau\beta$. Find all inital values $x(0)$ which are equilibria for this system.

Problem 7. Is the system ever linearly stable at the equilibrium points you found in Problem 6?

Problem 8. Show that $L(t) + C(t) + U(t)$ equals some constant κ independent of t. Suppose $\kappa > 0$ and $\alpha\gamma \neq \tau\beta$. Rewrite the system of differential equations as a system just involving L and C, using that $U = \kappa - L - C$. When you do this, which of the equilibria you bound in Problem 6 are stable for the two variable system involving only L and C? Your answer should depend on the α, β, τ and γ. Can you explain why this answer makes heuristic sense?

Remark Because there is a conserved quantity $L + C + U = \kappa$ which does not change with time, stability of the three variable system is not a natural condition. This is because such stability requires that all small variations of (L, C, U) from an equilibrium return toward the equilibrium, and most of these variations will not conserve $L + C + U$.

Extra Credit What happens in Problems 6 and 8 if $\alpha\gamma = \tau\beta$? You can turn this in at any time during the semester.