NOTES ON EPIDEMIC MODELS

T. CHINBURG, APRIL 13, 2020

1. The model

The goal of these notes is to set up a fairly general epidemic model which can be applied to a number of different situations.

The model has three different populations:

1. The "susceptible" population, of size $S(t)$ at time t.
2. The "infected" population, of size $I(t)$ at time t.
3. The "removed" population, of size $R(t)$ at time t.

Here are two examples.

Example 1.1. In the covid-19 pandemic, the "susceptible" population consists of people who are not now infected but who could become so. The "infected" population are those who are currently infected. The "removed" population could be the people who are currently immune to infection. Note that some of those currently immune might lose their immunity over time.

Example 1.2. The model will apply also to zombie epidemics. For example, suppose we say that someone has become zombified if they are unable to consider evidence which contradicts their prior beliefs. The "susceptible" population then consists of those people who have not become zombified, but who could become so at some point. This group is willing to consider new information, even if it contradicts their prior beliefs. The "infected" population consists of people who are zombified in the above sense. The "removed" population consists of those people who have removed themselves from interactions with other people over the controversy in question.

2. Dynamics of the model

We assume that there are positive constants $\alpha, \beta, \delta_1, \delta_2, \delta_3$ for which the following is true:

1. The susceptible and infected populations interact at a rate proportional to the product $S(t)I(t)$ of their numbers. The outcome of such an interaction can be no change, the infected could become removed, or the susceptible person could become infected. The latter conversions occur at rates $\beta S(t)I(t)$ and $\alpha S(t)I(t)$, respectively.
2. A removed person may be return to the infected population. The rate at this this occurs is proportional to $R(t)$ and it given by $\delta_1 R(t)$.
3. Members of the infected population may return to being susceptible at a rate $\delta_3 I(t)$ or become removed at rate $\delta_2 I(t)$.

Let's consider how to interpret the constants in this model in the two examples of the previous section.

2.1. **The covid-19 case.** It is reasonable to suppose that the rate of interactions between susceptible and infected people is on average proportional to the product of the sizes of their population. This leads to the term αSI in the rate at which susceptibles become infected.

At present, the interaction of a susceptible person with an infected person does have a certain probability of leading to the infected person recovering and becoming part of the "removed" group. For example, some fraction of the susceptible population consists of health care workers. We might assume that there rate at which interactions occur between infected people and health care workers is proportional to the product of the infected and susceptible populations. We assume in the model that such interactions lead with a certain rate per interaction to the infected individuals becoming removed (rather than immediately susceptible again). This leads to the term βSI of infected individuals becoming removed.

Infected individuals can spontaneously recover either to become susceptible or removed, at possibly different rates δ_3 and δ_2. Finally, we are supposing that removed individuals, who are currently not susceptible, may return to being infected at some rate δ_1.

2.2. **The zombie epidemic case.** We are supposing that the number of interactions per day between non-zombified and zombified individuals is proportional to the product of the sizes of their populations. Such interactions either result in the zombification of the non-zombified person, or the transfer of the zombified person to the "removed" population. In the classic zombie movie scenario, the latter occurs when a zombie is physically subdued and goes dormant, often only to return to a zombified state a short time later. The latter event results in the decay of members of the removed population back to the infected, zombie population.

The rate at which infected, zombified individuals decay back into being susceptible or removed are δ_3 and δ_1. In a typical zombie movie, one does not observe zombies spontaneously getting better and becoming susceptible but unzombified. However, politically zombified people interact less well with reality than those people who are susceptible to new information. Bad outcomes when coming into contact with reality may in fact lead to a small proportion of people with zombified views becoming willing to accept information that contradicts their prior beliefs. This phenomenon has certainly occurred during the covid-19 pandemic.

3. Differential equations

The autonomous differential equations which result from the model of the previous section are:

$$\frac{dS}{dt} = -\alpha SI + \delta_3 I \tag{3.1}$$

$$\frac{dI}{dt} = (\alpha - \beta)SI - (\delta_3 + \delta_2)I + \delta_1 R \tag{3.2}$$

$$\frac{dR}{dt} = \beta SI + \delta_2 I - \delta_1 R \tag{3.3}$$

Note that

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0.$$

So $S + I + R$ is constant, and we chooose to normalize this constant by

$$S(t) + I(t) + R(t) = 1 \quad \text{for all} \quad t \tag{3.4}$$

Thus S, I and R now represent the fractions of the total population represented by members of each group. One can thus consider SI to be the probability that a randomly chosen ordered pair of people will consist of first a susceptible and second an infected (or zombified) person.

Notice that (3.4) says that there are really only to independent functions in this system, rather than three. We can make this explicit in the following way. Substitute

$$R = 1 - (I + S)$$

into the above differential equations to have the two variable autonomous system of differential equations given by

$$\frac{dS}{dt} = -\alpha SI + \delta_3 I = G_1(S, I) \tag{3.5}$$

$$\begin{aligned} \frac{dI}{dt} &= (\alpha - \beta)SI - (\delta_3 + \delta_2)I + \delta_1(1 - (I + S)) \\ &= (\alpha - \beta)SI - (\delta_1 + \delta_2 + \delta_3)I - \delta_1 S + \delta_1 \\ &= G_2(S, I) \end{aligned} \tag{3.6}$$

If we let $f : \mathbb{R} \to \mathbb{R}^2$ be

$$f(t) = \begin{pmatrix} S(t) \\ I(t) \end{pmatrix}$$

and

$$G(S, I) = \begin{pmatrix} G_1(S, I) \\ G_2(S, I) \end{pmatrix}$$

then the system of differential equations is

$$\frac{df}{dt}(t) = G(f(t)). \tag{3.7}$$

4. Lecture 1 of "The Mathematical Modeling of Epidemics" by M. Iannelli

The model discussed in equation (4) of [1] amounts to these equations:

$$\frac{dS}{dt} = -\alpha SI \tag{4.8}$$

$$\frac{dI}{dt} = \alpha SI - \delta_2 I \tag{4.9}$$

$$\frac{dR}{dt} = \delta_2 I \tag{4.10}$$

For further interpretation of the constant α, see equations (1) and (2) of [1]. Recall that we are normalizing the constant $N = S(t) + I(t) + R(t)$ to be 1. This is not done in [1], so one has to take this into account when translating between these notes and [1]. We are focusing on the fraction $S(t)$ of the total population which is susceptible, for example, rather than on the absolute number of susceptibles.

The above specialization of the equations of the previous section amounts to the following additional hypotheses.

A. Interactions between susceptible individuals and infected individuals don't lead to infected individuals becoming removed, i.e. $\beta = 0$ in the previous section.
B. Infected individuals never spontaneously become susceptible again.
C. Removed individuals never become infected again.

This specialization of the prior model does not cover the typical zombie movie scenario very well, in which zombies who have been forced into a removed state spontaneously return to being zombies.

Anyone who has seen a typical zombie movie knows that usually, zombies triumph. However, we will see this is not the case in the above specialization. We now recap the derivation of the following result given in [1]:

Theorem 4.1. *Suppose $\alpha > 0$ and $\delta_2 > 0$. Let $S(t)$, $I(t)$ and $R(t)$ satisfy the differential equations (4.8), (4.9) and (4.10), and suppose*

$$1 = S(0) + I(0) + R(0)$$

with $S(0) > 0$, $I(0) > 0$ and $R(0) \geq 0$. Then $S(t)$ decreases monotonically as $t \to \infty$ to a positive limit S_∞ which satisfies the equation

$$S_\infty - \frac{\delta_2}{\alpha} \ln S_\infty = S(0) + I(0) - \frac{\delta_2}{\alpha} \ln S(0). \tag{4.11}$$

Furthermore, $\lim_{t\to\infty} I(t) = 0$ and $I(t) > 0$ for all $t \geq 0$.

Proof. From (4.8) and (4.9) we get

$$\frac{dS}{dt} \leq 0 \quad \text{and} \quad \frac{d(S+I)}{dt} = -\delta_2 I \leq 0 \tag{4.12}$$

Since $0 \leq S(t) \leq 1$, we conclude that $S(t)$ has non-increasing limit $S_\infty = \lim_{t\to\infty} S(t)$ as $t \to \infty$. We get from (4.10) that

$$0 \leq \delta_2 \int_0^t I(s)ds = R(t) - R(0) = (1 - S(t) - I(t)) - (1 - S(0) - I(0)) = S(0) - S(t) + I(0) - I(t) \leq 1.$$

Now $I(s) \geq 0$ for all s means $\delta_2 \int_0^t I(s)ds$ cannot decrease as t increases, so since we have shown this integral is bounded, we see that $\delta \int_0^\infty I(s)ds$ converges. So

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} S(0) - S(t) + I(0) - \int_0^t I(s)ds = S(0) - S_\infty + I(0) - \int_0^\infty I(s)ds$$

is well defined. However, this forces

$$\lim_{t\to\infty} I(t) = 0$$

because the integral $\int_0^\infty I(s)ds$ converges.

We now use the fact that (4.8) and (4.9) imply

$$\frac{d}{dt}(S + I - \frac{\delta_2}{\alpha}\ln S) = -\alpha SI + \alpha SI - \delta_2 I - \frac{\delta_2}{\alpha}\frac{dS/dt}{S} = 0.$$

So $S + I - \frac{\delta_2}{\alpha}\ln S$ is constant, and this gives (4.11).

Finally we need to check that S_∞ are $I(t)$ are positive. From (4.8) we get

$$\int_0^\infty \frac{d}{ds}\ln S ds = \int_0^\infty -\alpha I(s)ds$$

converges. This implies $\ln S(t)$ has a finite limit as $t \to \infty$, so S_∞ must be positive. From (4.9) we get

$$\frac{d}{dt}\ln I = \alpha S - \delta_2$$

So

$$\ln I(t) - \ln I(0) = \int_0^t \frac{d}{ds}\ln I ds = \int_0^t (\alpha S(s) - \delta_2)ds$$

is finite for all $t \geq 0$ and we conclude that $I(t)$ can never be 0. Now (4.8) shows $S(t)$ is monotonically decreasing, rather than just being non-increasing. □

Theorem 4.2. *If $S(0) < \delta_2/\alpha$, then $I'(t) < 0$ for all $t > 0$. Suppose now that $S(0) > \delta_2/\alpha$. There will be a unique time t^* such that $S(t^*) = \delta_2/\alpha$. One has in this case that $I'(t) > 0$ for $t < t^*$ and $I'(t) < 0$ for $t > t^*$.*

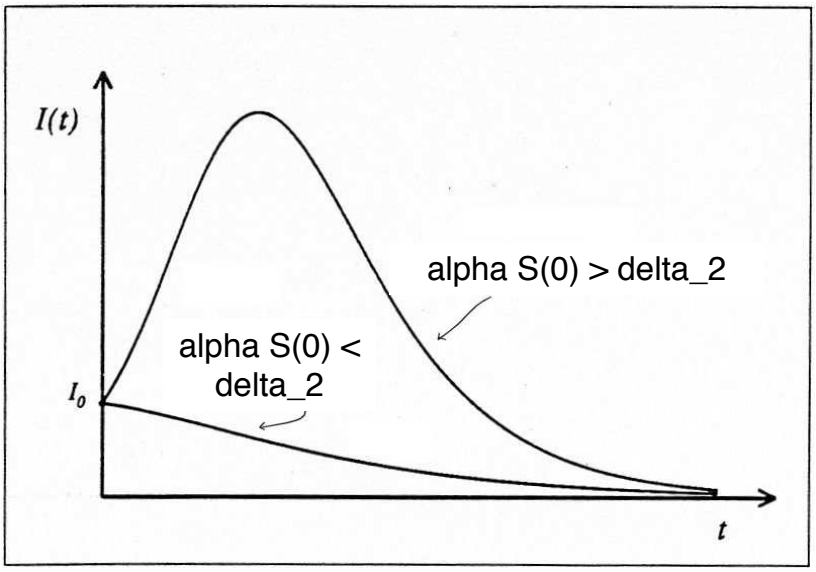
Proof. From (4.9) we get

$$\frac{dI}{dt} = (\alpha S - \delta_2)I \tag{4.13}$$

Since we showed $I(t) > 0$ of for all t and $S(t)$ is non-increasing with t, we see that $\frac{dI}{dt} < 0$ for all t if $\alpha S(0) < \delta_2$. Suppose now that $S(0) > \delta_2/\alpha$. Since we showed $\lim_{t\to\infty} I(t) = 0$, $\frac{dI}{dt}$ has to be negative for some value of t. Since we showed $S(t)$ is monotonically decreasing, we conclude from (4.13) that there will be unique t^* such that $\alpha S(t^*) - \delta_2 = 0$. The final statement in the Theorem is now clear from (4.13) and $I(t) > 0$ for all t.

□

Remark 4.3. The significance of Theorem 4.2 is that if $S(0) < \delta_2/\alpha$, there is a steady decline in the infected population, while if $S(0) > \delta_2/\alpha$ this population increases for awhile and then decreases. The peak happens when $t = t^*$, which is the time of the height of the resulting epidemic. This accounts for the difference between the steadily declining infected population curve and the peaked infected population curve appearing in the following diagram, which is Figure 8 of [1].



It's important to have a more explicit intepretation of the parameters α and δ_2 in order to estimate what values correspond to real world situations.

Lemma 4.4. *The exponential growth rate of the proportion $I(t)$ of the population which is infected is*

$$\frac{d \ln I}{dt} = \frac{dI/dt}{I} = (\alpha S - \delta_2) \tag{4.14}$$

If the time t is measured in units of days, this represents the net increase in the number of infected patients per day produced by each infected patient when one takes into account two processes. The first is new infections of susceptibles, leading to the term αS; the second is the transition rate of infected patients to the removed population, leading to the term δ_2. The units of $\frac{d \ln I}{dt}$ are in $1/days$ because these units correspond to change in the number of infected individuals per day per infected individual.

We can estimate αS by considering the situation in which S is very close to 1, so that nearly all the population is susceptible, and newly infected patients stay infected (so $\delta_2 = 0$). Thus α represents the number of newly infected individuals per day produced by one infection at the very beginning of an outbreak. One can estimate δ_2 by the rate at which infected individuals are removed by recovering or passing away.

Assuming that $S(0)$ is nearly 1, one will have an epidemic from the introduction of a small initial proportion $I(0)$ of infectives if $\delta_2 < \alpha$, so $1 \cong S(0) > \delta_2/\alpha$. If $\delta_2 > \alpha$, the appearance of a small proportion of infectives will be damped, with $I(t)$ decreasing monotonically to 0.

If T is the average time required for an infected individual to either recover or pass away, one can estimate δ_2 by $1/T$, and α_2 is approximately

$$\frac{\text{new infections caused by one infected person over } T \text{ days}}{T}.$$

This leads to the following simple corrollary

Corollary 4.5. *The constant δ_2/α is simply* 1 *divided by the number of new infections caused over T days by one infected person. One has an epidemic if the latter number is larger than* 1.

Example 4.6. To estimate δ_2 for the covid-19 pandemic, one needs to know how long on average it takes patients to either recover or pass away. Most cases (about 80%) don't require hospitalization, but some take up to 4 weeks to resolve one way or another. One could determine δ_2 from finding such data. For simplicity, let's estimate 2 weeks as also the average time for a case to resolve, corresponding to $T = 14$ days. Early esimates of the infectiousness of covid-19 were that without social distancing measures, each new case would produce 2.5 new cases over a two week period. This would correspond to $\alpha = 2.5/T = 2.5/14$ in units of 1/days. To estimate δ_2, one can ask how long on average it takes for patients to either recover or pass away. Most cases (about 80%) don't require hospitalization, but some take up to 4 weeks to resolve one way or another. One could determine δ_2 from finding such data. For simplicity, let's estimate 2 weeks as also the average time for a case to resolve. This leads to $\delta_2 = 1/14$ in units of 1/days. The critical value for S is

$$S(t^*) = \delta_2/\alpha = 1/2.5 = 0.40$$

where t^* is the time of when the proportion $I(t^*)$ of the population which is infected will be at its peak. This means that at the peak of the epidemic, only 40% of the population will be still be susceptible. We can estimate the maximal infected proportion $I(t^*)$ of the population using the fact that $F = S + I - \frac{\delta_2}{\alpha} \ln S$ is constant because its derivative as a function of t is 0. Taking $S(0) \cong 1$ and $I(0) \cong 0$ gives

$$S(t^*) + I(t^*) - \frac{\delta_2}{\alpha} \ln S(t^*) = S(0) + I(0) - \frac{\delta_2}{\alpha} \ln S(0) \cong 1.$$

Using $S(t^*) = \frac{\delta_2}{\alpha}$ we get

$$I(t^*) = 1 - \frac{\delta_2}{\alpha} \cdot (1 - \ln(\frac{\delta_2}{\alpha})) \tag{4.15}$$

When $\delta_2/\alpha = 0.40$ this leads to

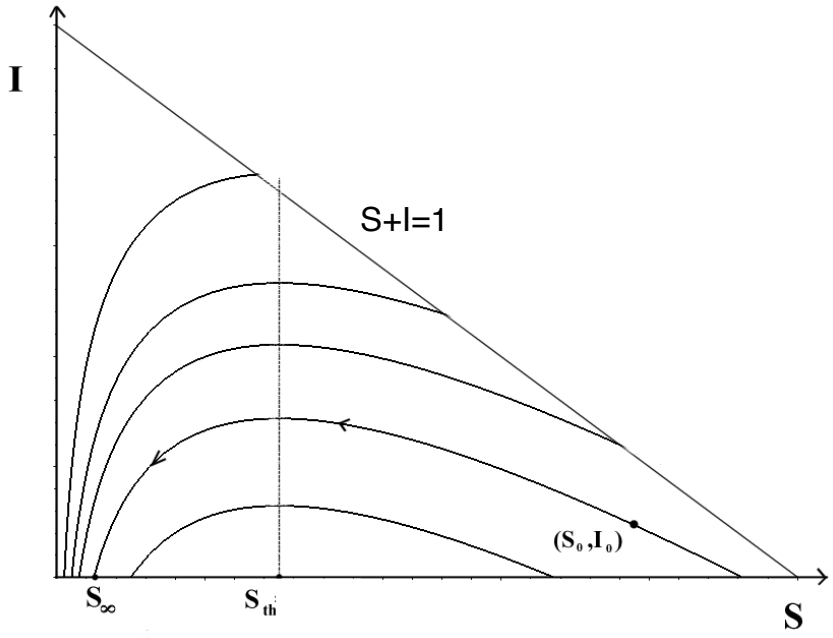
$$I(t^*) = 0.233484...$$

Thus 23% of the population will be infected at the peak of the epidemic, and $R(t^*) = 1 - S(t^*) - I(t^*) = 0.366516$ will be the fraction of the population that has been removed with by recovery or by passing away. At the end of the epidemic, the fraction S_∞ will satisfy

$$S_\infty - \frac{\delta_2}{\alpha} \ln S_\infty = S(0) + I(0) - \frac{\delta_2}{\alpha} \ln S(0) \cong 1$$

which leads to the numerical value $S_\infty = 0.107355$. Thus only 10.7% of the population will remain susceptible, leading to 90.3% of the population having been removed. In view of the fact that the removed population represents people who have either recovered or passed away, a fatality rate of p would lead to $0.903 \cdot p$ as the proportion of the population which passed away. Given that p seems to be around 0.01, this would mean $0.903 \cdot 0.01 = 0.00903$ of the population would have passed away, or around $328 \cdot 0.00903 = 2.96$ million people in the U.S. alone. This is why alarm bells went off at the CDC and WHO once the parameters δ_2, α and p became more certain.

We showed that $F = S + I - \frac{\delta_2}{\alpha} \ln S$ is constant because its derivative as a function of t is 0. So as t varies, $(S(t), I(t))$ moves along one of the level curves of F, as pictured below in the (S, I)-plane. $S_{th} = \delta_2/\alpha$ is the value of S where I is maximal along a given level curve.



If one starts with a value for $S(0)$ that is larger than $S_{th} = \delta_2/\alpha$, $I(t)$ increases for small t to a maximum at time t^* with $S(t^*) = S_{th} = \delta_2/\alpha$, and then $I(t)$ decreases to 0.

If one starts at a value of $S(0)$ less than $S_{th} = \delta_2/\alpha$, then $I(t)$ is monotonically decreasing. So, for instance, one could start at a point on one of the level curves of F which is to the left of the vertical line where $S = S_{th}$. Then as shown by the level curves, $I(t)$ is driven monotonically down toward 0, and $S(t)$ approaches a limit S_∞ to the left of S_{th}.

A key question is what happens when one starts initially with 0 infections and then a small number of infections enter the population. This corresponds to starting a a point on the S-axis in the last diagram and then moving slightly above the S-axis.

Suppose, for example, that $S(0) > S_{th}$ and $I(0)$ is close to 0. Then $I(t)$ increases while $S(t)$ decreases, leading to an epidemic. In this case S_∞ could end up being far smaller than $S(0)$ due to the fact that ends up on the other end of a the level curve in the (S, I) plane through the stating point $(S(0), I(0))$.

On the other had, if $S(0) < S_{th}$, then both infectives and susceptibles decrease, with infectives decreasing monotonically to 0. If $I(0)$ is very small, we can estimate by how much $S(0)$ decreases using

$$S_\infty - \frac{\delta_2}{\alpha} \ln S_\infty = S(0) + I(0) - \frac{\delta_2}{\alpha} \ln S(0).$$

Rewrite this relationship as

$$(S_\infty/S_0 - 1)S_0 - \frac{\delta_2}{\alpha}\ln(S_\infty/S_0) = I(0).$$

Write $S_\infty/S(0) = 1 - \epsilon$ for a small $\epsilon > 0$, and approximate $\ln(S_\infty/S_0)$ by $-\epsilon$. Then

$$-\epsilon S_0 - \frac{\delta_2}{\alpha}(-\epsilon) = I(0)$$

so

$$\epsilon = I(0)/(\frac{\delta_2}{\alpha} - S_0) = I(0)/(S_{th} - S(0)).$$

So the amount by which the susceptibles decrease is approximately

$$S(0) - S_\infty = S(0)\cdot\epsilon = I(0)\cdot\frac{S(0)}{S_{th} - S(0)} \tag{4.16}$$

Another way to view this is that when when $S(0) < S_{th}$, each member of small group of infectives introduced into an otherwise healthy population ends up decreasing the number of susceptibles by about $\frac{S(0)}{S_{th}-S(0)}$ people in the course of the disease running its course. The outcome is very different when $S(0) > S_{th}$.

One issue that has come up recently is the possibility of successive waves of appearances of covid-19 should social distancing measures be abated. The above model has something interesting to say about this. Suppose that α and δ_2 are fixed. One might start with an epidemic condition, in which $S(0) > S_{th} = \delta_2/\alpha$. In the limit as $t \to \infty$ $S(t)$ approaches a value S_∞which is less than S_{th} and $\lim_{t\to\infty} I(t) = 0$. Suppose that some large t, some new infectives are introduced from another source, increasing $I(t)$. For large time, $S(t)$ will be less than S_{th}, so slightly increasing $I(t)$ can lead only to damped solution in which $I(t)$ decreases monotonically to 0.

In other words, without a change in α and δ_2, once the initial wave of the infection has passed, there will not be enough susceptibles left to produce another epidemic if some additional infectives are introduced. Instead, the new infection will die out monotonically. This does not contradict current models of what may happen with covid-19 if social distancing lessens. Such lessening would change the values of α and δ_2, leading to a time evolution on a different set of level curves. We can say that if one maintains social distancing, this model predicts success and an absence of new epidemics later unless covid-19 mutates in such a way as to change the parameters of the model.

5. Autonomous ordinary differential equations

5.1. **Iannelli's example.** To introduce this topic, let's start with the model of Iannelli discussed in the previous section. This is determined by the proportions $S(t)$ and $I(t)$ of the total population which are susceptible and infected, respectively. Two basic questions are:

A. Which initial values $S(0)$ and $I(0)$ have the property that setting $S(t) = S(0)$ and $I(t) = I(0)$ for all t give solutions of the differential equations defining the model. These are **equilibrium solutions**.
B. Suppose $(S(0), I(0))$ does define an equilibrium solution. If we vary $S(0)$ or $I(0)$ slightly, will the system always evolve in time back to the original equilibrium? Such equilibria are called **stable**.

The relevance of stable equilibria is that in the real world, one can expect initial conditions to vary somewhat. So only the stable equilibria at likely to be seen for any length of time.

Let's answer these questions for the system discussed in the previous section before formulating stability questions in general.

At an equilibrium, both $\frac{dS}{dt}$ and $\frac{dI}{dt}$ must be 0 for all t. So (4.8) and (4.9) give

$$-\alpha SI = \alpha SI - \delta_2 I = 0.$$

Since we assumed $\alpha, \delta_2 > 0$, this forces $I(t) = 0$ for all time. Setting $S(t) = S(0)$ for all t then gives an equilibrium solution.

Suppose now that we start with an equilbrium solution, so $S(t) = S(0)$ for all t and $I(t) = 0$, for all t. This equilibrium is stable if and only for all sufficiently small perturbations of the initial conditions, the systems returns in the limit as $t \to \infty$ to the original equilibrium. In particular, this would have to be the case that if we hold $I(0)$ equal to 0 but change $S(0)$ slightly. This leads to another equilibrium, which does not return over time to the original one. So no equilibrium is stable.

In practical terms, this means that for a disease modelled by the system of the previous section, one can't expect that small variations in initial conditions will always lead back to exactly the same conditions. However, we did see in the past section that there is a qualitative difference in the changes produced by the introduction of a small number of infectives into an otherwise healthy popluation when $S(0) > \delta_2/\alpha$ (epidemic condiions) and when $S(0) < \delta_2/\alpha$ (damped infections). The former can lead to large changes in the number of susceptibles relative to the number of infectives introduced, while the latter leads to changes in the number of susceptibles that are bounded in terms of number of infectives by estimates of the form (4.16).

5.2. **Autonomous linear differential equations, equilibria and stability.** Suppose that $n \geq 1$. We are going to later multiply vectors of length n on the left by square matrices. So we will consider vectors of length n to be column vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}.$$

This will lessen the need to take transposes of row vectors as in the lectures in class.

Let $G : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable vector valued function. Thus G assigns to each $x \in \mathbb{R}^n$ a vector

$$G(x) = \begin{pmatrix} G_1(x) \\ G_2(x) \\ \cdots \\ G_n(x) \end{pmatrix}$$

and all the partial derivatives $\frac{\partial G_i}{\partial x_j}$ are continuous functions of x.

Definition 5.1. An autonomous ordinary differential equation for a function $f : \mathbb{R} \to \mathbb{R}^n$ has the form

$$\frac{df}{dt}(t) = G(f(t)) \quad \text{and} \quad f(0) = x \tag{5.17}$$

in which $f : \mathbb{R} \to \mathbb{R}^n$ is a function and x is an initial value for f. Here

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \cdots \\ f_n(t) \end{pmatrix}$$

for some functions $f_1, \ldots, f_n$ from $\mathbb{R}$ to $\mathbb{R}$ and

$$\frac{df}{dt}(t) = \begin{pmatrix} \frac{df_1}{dt}(t) \\ \\ \frac{df_2}{dt}(t) \\ \\ \cdots \\ \frac{df_n}{dt}(t) \end{pmatrix}.$$

The differential equation (5.17) is called autonomous because $\frac{df}{dt}$ depends only on what $f(t)$ is rather than depending on both $f(t)$ and the independent variable t. If one thinks of t as time and $f(t)$ as the state of a system at time t, then the rate of change $\frac{df}{dt}$ of the system depends only on its state $f(t)$ at time t, not on what time it is.

Definition 5.2. The initial value x gives an equilibrium solution if $f(t) = x$ for all t is a solution of the differential equation (5.17).

The following result is easy:

Lemma 5.3. *The vector x gives an equilibrium solution $f(t) = x$ for all t if and only if $G(x)$ is the zero vector $\underline{0} = (0, \ldots, 0)^{transpose}$.*

Proof. If $f(t) = x$ for all t is a solution to the differential equation, we have

$$\frac{df}{dt}(t) = \underline{0} = G(f(t)) = G(x) \quad \text{for all} \quad t. \tag{5.18}$$

Conversely, if $G(x) = \underline{0}$, and we let $f(t) = x$ for all t, then (5.18) holds, so $f(t)$ solves the differential equation. □

Definition 5.4. An equilibrium solution $f(t) = x$ for all t is stable if there an $\epsilon > 0$ such that the following is true for all $\tilde{x} \in \mathbb{R}^n$ such that distance$(x, \tilde{x}) < \epsilon$. Suppose $\tilde{f}(x)$ is a solution of the differential equation

$$\frac{d\tilde{f}}{dt}(t) = G(\tilde{f}(t))$$

with initial condition $\tilde{f}(0) = \tilde{x}$. Then

$$\lim_{t \to +\infty} \tilde{f}(t) = x.$$

5.3. **Linearizing the differential equation, and linear stability.** To study stability near an equilibrium x, we use the Taylor expansion of $G(\tilde{x})$ for $\tilde{x}$ near x. The Taylor expansion is

$$G(\tilde{x}) = G(x) + \text{Jac}(G)(x) \cdot (\tilde{x} - x) + \quad \text{higher order terms.}$$

Here Jac(G) is the $n \times n$ matrix

$$\left(\frac{\partial G_i}{\partial x_j} \right)_{1 \le i,j \le n}$$

and $\mathrm{Jac}(G)(x)$ is the matrix of constants which results from evaluating at x all the partial derivatives which are the entries of $\mathrm{Jac}(G)$. The higher order terms in the expansion go to 0 more rapidly than any linear function of $\tilde{x} - x$ and $\tilde{x}$ tends toward x.

If $\tilde{f}(t)$ were a solution of the differential equation such that $\tilde{f}(t)$ is always close to x, then since $G(x) = \underline{0}$ for an equilibrium value of x, we would have

$$G(\tilde{f}(t)) = \mathrm{Jac}(G)(x) \cdot (\tilde{f}(t) - x) \quad + \quad \text{smaller order error.}$$

We use the first term on the right side of this expression to get a linear approximation to our original differential equation:

Definition 5.5. The linear approximation to the differential equation $\frac{d\tilde{f}}{dt}(t) = G(\tilde{f}(t))$ near an equilibrium value x is

$$\frac{d\tilde{f}}{dt}(t) = A \cdot (\tilde{f}(t) - x) \tag{5.19}$$

When A is the constant $n \times n$ matrix $A = \mathrm{Jac}(G)(x)$.

Lemma 5.6. *The initial value x is a equilibrium solution of (5.19), in the sense that defining $\tilde{f}(t) = x$ for all t is a solution of (5.19). Further, x is a stable equilibrium for (5.19) if and only if every solution $y(t)$ of the differential equation*

$$\frac{dy}{dt}(t) = Ay(t) \tag{5.20}$$

has the property that $\lim_{t\to+\infty} y(t) = \underline{0} = (0, \ldots, 0)^{transpose}$, where $A = \mathrm{Jac}(G)(x)$.

Proof. Clearly $\tilde{f}(t) = x$ for all t is a solution of (5.19), since then $\frac{d\tilde{f}}{dt}(t) = 0$.

Suppose first that all $y(t)$ as in the Lemma have $\lim_{t\to+\infty} y(t) = \underline{0}$. By definition, x is a stable equilibrium solution of (5.19) if and only for all $\tilde{f}(t)$ for which (5.19) holds and $\tilde{f}(0) = \tilde{x}$ is sufficiently close to x, we have $\lim_{t\to+\infty} \tilde{f}(t) = x$. Set $y(t) = \tilde{f}(t) - x$. Then $\frac{dy}{dt} = \frac{d\tilde{f}}{dt}$, so $y(t)$ is a solution of (5.20). Thus

$$\lim_{t\to+\infty} \tilde{f}(t) = \lim_{t\to+\infty} (y(t) + x) = \underline{0} + x = x$$

and (5.19) has x as a stable equilibrium solution.

Conversely, suppose that x is a stable equilibrium solution to (5.19), and that $y(t)$ is a solution of (5.20). By definition, this means that there is an $\epsilon > 0$ such that if $\mathrm{distance}(x, \tilde{x}) < \epsilon$, then every solution $\tilde{f}$ of (5.19) such that $\tilde{f}(0) = \tilde{x}$ has the property that

$$\lim_{t\to+\infty} \tilde{f}(t) = x.$$

We can find a real constant $c > 0$ such that $\mathrm{distance}(\underline{0}, c \cdot y(0)) < \epsilon$. Since $y(t)$ is a solution of (5.20), we find that $\tilde{f}(t) = x + c \cdot y(t)$ is a solution of (5.19) since

$$\frac{d\tilde{f}}{dt}(t) = c \cdot \frac{dy}{dt}(t) = c \cdot Ay(t) = Ac \cdot y(t) = A(\tilde{f}(t) - x).$$

We have

$$\mathrm{distance}(\tilde{f}(0), x) = \mathrm{distance}(x + c \cdot y(0), x) = \mathrm{distance}(cy(0), \underline{0}) < \epsilon.$$

Therefore $\lim_{t\to+\infty} \tilde{f}(t) = x$ since x is a stable equilibrium, so

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} (\tilde{f}(t) - x)/c = 0.$$

□

Definition 5.7. Let x be an equilibrium solution of the original differential equation (5.17). We will say that x is a linearly stable equilibrium if the linearized differential equation (5.5) has x as a stable solution. By Lemma 5.6, this is equivalent to requiring that $\lim_{t\to+\infty} y(t) = \underline{0}$ for all solutions $y(t)$ to

$$\frac{dy}{dt}(t) = Ay(t)$$

when $A = \mathrm{Jac}(G)(x)$.

5.4. **Solutions of linear systems.** Suppose A is an $n \times n$ matrix of complex numbers.

Theorem 5.8. *The series of $n \times n$ matrices*

$$e^{At} = \sum_{m=0}^{\infty} \frac{(At)^m}{m!}$$

converges for all values of t to an $n \times n$ matrix of complex numbers. The unique solution $y : \mathbb{R} \to \mathbb{R}^n$ of the differential equation

$$\frac{dy}{dt}(t) = Ay(t) \quad \text{and} \quad y(0) = w \tag{5.21}$$

is

$$y(t) = e^{At}w.$$

This result does take some time to prove rigorously, so I will not include a proof. The appearance of e^{At} is justified by the following formal computation:

$$\begin{aligned}
\frac{d}{dt}e^{At}w &= \frac{d}{dt}\sum_{m=0}^{\infty}\frac{A^m t^m}{m!}w \\
&= \sum_{m=0}^{\infty}\frac{d}{dt}\frac{A^m t^m}{m!}w \\
&= \sum_{m=1}^{\infty}\frac{A^m m t^{m-1}}{m!}w \\
&= \sum_{m=1}^{\infty}A\frac{A^{m-1}t^{m-1}}{(m-1)!}w \\
&= Ae^{At}w \qquad (5.22)
\end{aligned}$$

The following result is proved in linear algebra courses.

Theorem 5.9. *There is an invertible $n \times n$ matrix B such that $C = BAB^{-1}$ has the upper triangular form*

$$C = \begin{pmatrix} \lambda_1 & c_{1,2} & c_{1,3} & \cdots & c_{1,n} \\ 0 & \lambda_2 & c_{2,3} & \cdots & c_{2,n} \\ 0 & 0 & \lambda_3 & \cdots & c_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix}. \tag{5.23}$$

for some complex numbers $\lambda_1, \cdots, \lambda_n$ and $c_{i,j}$ with $i < j$. The entries below the diagonal of C are all 0. The numbers λ_i for $1 \le i \le n$ may not all be distinct, and are called

the eigenvalues of A. These λ_i are the roots (counting mulitiplicities) of the characteristic polynomial

$$c_A(T) = \det(T \cdot I_n - A)$$

where T is an indeterminate and I_n is the $n \times n$ identity matrix.

Observe now that if $m \geq 0$, then

$$C^m = (BAB^{-1}) = (BAB^{-1}) \cdot (BAB^{-1}) \cdots (BAB^{-1}) = BA^m B^{-1}$$

since we can group terms in the product to take advantage of the fact that $BB^{-1} = I_n$. Thus we get

$$\begin{aligned} Be^{At}B^{-1} &= B\left(\sum_{m=0}^{\infty} \frac{(At)^m}{m!}\right) B^{-1} \\ &= \sum_{m=0}^{\infty} \frac{BA^m B^{-1} t^m}{m!} \\ &= \sum_{m=0}^{\infty} \frac{C^m t^m}{m!} \\ &= e^{Ct} \end{aligned} \tag{5.24}$$

Since C in (5.23) is upper triangular, the matrix C^m is also upper triangular, with diagonal terms given by $\lambda_1^m, \ldots, \lambda_n^m$. Thus

$$\begin{aligned} e^{Ct} &= \sum_{m=0}^{\infty} \frac{C^m t^m}{m!} \\ &= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{\lambda_1^m t^m}{m!} & * & * & \cdots & * \\ 0 & \sum_{m=0}^{\infty} \frac{\lambda_2^m t^m}{m!} & * & \cdots & * \\ 0 & 0 & \sum_{m=0}^{\infty} \frac{\lambda_3^m t^m}{m!} & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \sum_{m=0}^{\infty} \frac{\lambda_n^m t^m}{m!} \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda_1 t} & * & * & \cdots & * \\ 0 & e^{\lambda_2 t} & * & \cdots & * \\ 0 & 0 & e^{\lambda_3 t} & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & e^{\lambda_n t} \end{pmatrix} \end{aligned} \tag{5.25}$$

where we don't need to calculate the entries above the diagonal.

Suppose now that x is a linearly stable equilibrium solution of the original differential equation (5.17) in the sense of Definition 5.7. Let $A = \mathrm{Jac}(G)(x)$. Then for all initial values $y(0) = w$ of the differential equation

$$\frac{dy}{dt}(t) = Ay(t)$$

we should have $\lim_{t \to +\infty} y(t) = 0$. By Theorem 5.8, this is equivalent to

$$\lim_{t \to +\infty} e^{At} w = \underline{0}.$$

Taking w to range over the vectors which have exactly one component equal to 1 and all the others equal to 0, we would be able to conclude that

$$\lim_{t\to+\infty} e^{At} = \text{the zero matrix.}$$

Then (5.24) would give

$$\lim_{t\to+\infty} e^{Ct} = B\left(\lim_{t\to+\infty} e^{At}\right)B^{-1} = \text{the zero matrix.}$$

But now (5.25) would show

$$\lim_{t\to+\infty} e^{\lambda_i t} = 0 \tag{5.26}$$

for $i = 1, \ldots, n$. Here if $\lambda_i = a_i + b_i\sqrt{-1}$ we have

$$e^{\lambda_i t} = e^{a_i t}(\cos(b_i t) + \sqrt{-1}\sin(b_i t)).$$

So (5.26) is equivalent to

$$\mathrm{Re}(\lambda_i) = a_i < 0 \quad \text{for} \quad i = 1, \ldots, n. \tag{5.27}$$

With a little more work, which I won't include, one can show that (5.27) is also sufficient for linear stability:

Theorem 5.10. *The differential equation (5.17) is linearly stable at an equilibrium x if and only if the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the Jacobian matrix $A = \mathrm{Jac}(G)(x)$ at x have all negative real parts.*

6. Stability analysis when recovered individuals can return to being infected

We now return to the original epidemic model discussed in §2, but we assume a positive rate for recovered individuals to return to being infected. This is certainly the case in standard zombie epidemics. Depending on how they were removed, zombies have the abillity to return to being infective after being removed for some time. In terms of covid-19, it appears that people who appear to have recovered have a small probability of becoming infected again. We will see that this possibility radically changes the dynamics of the model.

Hypothesis 6.1. The constant δ_1 representing the rate at which recovered individuals return to being infected is positive. Futhermore α and δ_2 are both positive, while $\beta \geq 0$ and $\delta_3 \geq 0$.

Theorem 6.2. *There are two possible equilibria for the two population system (3.7):*

1. *There is an equilibrium at $x_0 = \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in which $R = 0$. This "all susceptible" equilibrium is linearly stable if and only if $\alpha < \delta_3$.*
2. *If $\alpha > \delta_3$ there is an additional "partial susceptible" equilibrium in which $S = \delta_3/\alpha < 1$. The values of I and R at this equilibrium are given by (7.32) and (7.33) below. This equilibrium is linearly stable.*

We can interpret the requirement that $\alpha > \delta_3$ in order for there to be an equilibrium in which S is not 1 in the following way.

A. In the case of the covid-19 epidemic, α is the rate at which encounters between susceptible individuals and infected individuals lead to new infections. The constant δ_3 in this case is the rate at which infected people spontaneously cease being infected, but return to being susceptible.
B. In the case of epidemics of people whose views become zombified, α measures the rate at which encounters between susceptible people and zombies lead to the susceptible person becoming zombified. The constant δ_3 measures the decay rate of zombies back into susceptibles, e.g. because encounters with reality tend to not go well for zombies.

In these scenarios, when $\alpha > \delta_3$, it makes sense in this case that not everyone will be in the susceptible group at the equilibrium. The proportion of susceptibles in the general population at equilibrium is

$$S = \frac{\delta_3}{\alpha} = \frac{\text{decay rate of infected individuals into susceptibles}}{\text{success rate of infected individuals infecting susceptibles}}.$$

Notice that if $\alpha > \delta_3$, then there will still be the "all susceptible" equilibrium in part (1) of the theorem. However, it will be unstable, so the introduction of even a few infected individuals will move the population toward the equilibrium in which $S = \delta_3/\alpha < 1$.

7. Computation of equilibria

A vector $x_0 = \begin{pmatrix} S \\ I \end{pmatrix}$ is an equilibrium of (3.7) if

$$G(x_0) = \begin{pmatrix} G_1(S,I) \\ G_2(S,I) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

since then $f(t) = x_0$ for all t is a solution.

In view of (4.8), the solutions of

$$G_1(S,I) = -\alpha SI + \delta_3 I = (-\alpha S + \delta_3)I$$

must have

$$I = 0 \quad \text{or} \quad -\alpha S + \delta_3 = 0. \tag{7.28}$$

7.1. The all susceptible equilibrium.

Suppose $I = 0$. Then

$$0 = G_2(S,I) = (\alpha - \beta)SI - (\delta_1 + \delta_2 + \delta_3)I - \delta_1 S + \delta_1 = \delta_1(1 - S)$$

forces $S = 1$. Since $S + I + R = 1$ we get $R = 0$. So this equilibrium has $(S, I, R) = (1, 0, 0)$ and

$$x_0 = \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

7.2. The partial susceptible equilibrium.

In view of (7.28), the other possible equilibrium has

$$S = \frac{\delta_3}{\alpha} \tag{7.29}$$

Notice that since $S+I+R = 1$ and each of S, I and R must be non-negative, this equilibrium makes physical sense only if

$$\delta_3 \leq \alpha \tag{7.30}$$

If $\delta_3 > \alpha$ then the only equilbrium which is physically meaningful is when $(S, I, R) = (1, 0, 0)$. This makes some sense, since if $\delta_3 > \alpha$ then the decay of infected people back

into susceptibles overwhelms the rate at which susceptibles become infected, resulting in the asymptotic disappearance of infections as $t \to \infty$.

We are going to prove the following:

Theorem 7.1. *If $\delta_3 \leq \alpha$ there is an equilibrium for the system (3.7) in which*

$$x_0 = f(0) = \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} \frac{\delta_3}{\alpha} \\ \\ \frac{\delta_1(1-\frac{\delta_3}{\alpha})}{\delta_1+\frac{\beta}{\alpha}\delta_3+\delta_2} \end{pmatrix}$$

In this case,

$$R = 1 - (S + I) = 1 - \frac{\delta_3}{\alpha} - \frac{\delta_1(1 - \frac{\delta_3}{\alpha})}{\delta_1 + \frac{\beta}{\alpha}\delta_3 + \delta_2}$$

and all of S, I and R lie in the interval $[0, 1]$.

It is interesting that $I \to 0$ as $\delta_1 \to 0$. So without the reinfection of members of the removed group, the infected population tends toward zero in this equilibrium, and R tends toward $1 - S = 1 - \frac{\delta_3}{\alpha}$.

To begin the proof, we need to calculate I and R in case (7.29) and (7.30) hold. We have

$$\begin{aligned} 0 &= G_2(S, I) \\ &= (\alpha - \beta)SI - (\delta_1 + \delta_2 + \delta_3)I - \delta_1 S + \delta_1 \\ &= I \cdot ((\alpha - \beta)S - (\delta_1 + \delta_2 + \delta_3)) + \delta_1(1 - S) \\ &= I \cdot ((\alpha - \beta)\frac{\delta_3}{\alpha} - (\delta_1 + \delta_2 + \delta_3)) + \delta_1(1 - \frac{\delta_3}{\alpha}) \\ &= I \cdot ((1 - \frac{\beta}{\alpha})\delta_3 - (\delta_1 + \delta_2 + \delta_3)) + \delta_1(1 - \frac{\delta_3}{\alpha}) \\ &= -I(\delta_1 + \frac{\beta}{\alpha}\delta_3 + \delta_2) + \delta_1(1 - \frac{\delta_3}{\alpha}) \end{aligned} \tag{7.31}$$

This leads to

$$I = \frac{\delta_1(1 - \frac{\delta_3}{\alpha})}{\delta_1 + \frac{\beta}{\alpha}\delta_3 + \delta_2} \tag{7.32}$$

Note that we have assumed (7.30) in this case, so $0 \leq 1 - \frac{\delta_3}{\alpha} < 1$ since we have assumed all the constants $\alpha, \beta, \delta_1, \delta_3, \delta_2$ are positive. So we get

$$0 \leq \delta_1(1 - \frac{\delta_3}{\alpha}) < \delta_1 < \delta_1 + \frac{\beta}{\alpha}\delta_3 + \delta_2.$$

It follows from (7.32) that at this equilibrium,

$$0 \leq I < 1$$

so this value for I is physically meaningful.

Finally, we have

$$\begin{aligned} R &= 1 - (S + I) \\ &= 1 - \frac{\delta_3}{\alpha} - \frac{\delta_1(1 - \frac{\delta_3}{\alpha})}{\delta_1 + \frac{\beta}{\alpha}\delta_3 + \delta_2} \end{aligned} \tag{7.33}$$

Clearly $R \leq 1$. We should check when $0 \leq R$. Clearly $0 \leq R$ if and only if

$$\frac{\delta_3}{\alpha} + \frac{\delta_1(1 - \frac{\delta_3}{\alpha})}{\delta_1 + \frac{\beta}{\alpha}\delta_3 + \delta_2} \leq 1. \tag{7.34}$$

Here

$$\frac{\delta_1}{\delta_1 + \frac{\beta}{\alpha}\delta_3 + \delta_2} < 1$$

so we have

$$\frac{\delta_3}{\alpha} + \frac{\delta_1(1 - \frac{\delta_3}{\alpha})}{\delta_1 + \frac{\beta}{\alpha}\delta_3 + \delta_2} < \frac{\delta_3}{\alpha} + (1 - \frac{\delta_3}{\alpha}) = 1.$$

We conclude that the equilibrium (S, I, R) given by (7.29), (7.32) and (7.33) is physically meaningful provided (7.30) holds, i.e. provided $\delta_3 \leq \alpha$. The latter requirement says that the decay rate of infected individuals into susceptibles is smaller that the rate at which susceptibles become infected.

8. Stability analysis

We now apply the techniques discussed in class to studying the linear stability of each equilibrium $x_0 = \begin{pmatrix} S \\ I \end{pmatrix}$ analyzed in previous section. Recall that x_0 is linearly stable if and only if the eigenvalues of $Jac(G)(x_0)$ all have negative real part. Here

$$G(S, I) = \begin{pmatrix} G_1(S, I) \\ G_2(S, I) \end{pmatrix} = \begin{pmatrix} (-\alpha S + \delta_3)I \\ (\alpha - \beta)SI - (\delta_1 + \delta_2 + \delta_3)I - \delta_1 S + \delta_1 \end{pmatrix}.$$

The Jacobian $Jac(G)$ is

$$Jac(G) = \begin{pmatrix} \frac{\partial G_1}{\partial S} & \frac{\partial G_1}{\partial I} \\ \frac{\partial G_2}{\partial S} & \frac{\partial G_2}{\partial I} \end{pmatrix} = \begin{pmatrix} -\alpha I & -\alpha S + \delta_3 \\ (\alpha - \beta)I - \delta_1 & (\alpha - \beta)S - (\delta_1 + \delta_2 + \delta_3) \end{pmatrix} \tag{8.35}$$

8.1. **The all susceptible equilibrium.** Suppose $(S, I, R) = (1, 0, 0)$. Then (8.35) for $x_0 = \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives

$$A = Jac(G)(x_0) = \begin{pmatrix} 0 & -\alpha + \delta_3 \\ -\delta_1 & (\alpha - \beta) - (\delta_1 + \delta_2 + \delta_3) \end{pmatrix} \tag{8.36}$$

We showed in class that a two-by-two matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with real coefficients has the property that its eigenvalues have negative real part if and only if

$$\mathrm{Det}(A) = ad - bc > 0 \quad \text{and} \quad \mathrm{Trace}(A) = a + d < 0. \tag{8.37}$$

Here

$$\mathrm{Det}(Jac(G)(x_0)) = -(-\alpha + \delta_3) \cdot (-\delta_1) = \delta_1(\delta_3 - \alpha) \tag{8.38}$$

Since $\delta_1 > 0$, we conclude that

$$\mathrm{Det}(Jac(G)(x_0)) > 0 \quad \text{if and only if} \quad \delta_3 > \alpha \tag{8.39}$$

We have

$$\text{Trace}(Jac(G)(x_0)) = (\alpha - \beta) - (\delta_1 + \delta_2 + \delta_3) \tag{8.40}$$

Here all of $\alpha, \beta, \delta_1, \delta_3, \delta_2$ are positive. So if $\text{Det}(Jac(G)(x_0)) > 0$, we have $\alpha - \delta_3 < 0$ and we get

$$\text{Trace}(Jac(G)(x_0)) = (\alpha - \delta_3) - (\delta_1 + \beta + \delta_2) < 0.$$

The conclusion is:

Theorem 8.1. *The equilibrium* $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *is linearly stable if and only if* $\delta_3 > \alpha$.

8.2. **The partial susceptible equilibrium.** We now suppose that we at the equilibrium (S, I, R) given by (7.29), (7.32) and (7.33). We will also assume that this is physically meaningful, so that (7.30) holds, i.e. $\delta_3 \leq \alpha$. The Jacobian matrix at

$$x_0 = \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} \frac{\delta_3}{\alpha} \\ \\ \frac{\delta_1(1-\frac{\delta_3}{\alpha})}{\delta_1 + \frac{\beta}{\alpha}\delta_3 + \delta_2} \end{pmatrix}$$

is

$$\begin{aligned} A &= Jac(G)(x_0) \\ &= \begin{pmatrix} -\alpha I & -\alpha S + \delta_3 \\ \\ (\alpha - \beta)I - \delta_1 & (\alpha - \beta)S - (\delta_1 + \delta_2 + \delta_3) \end{pmatrix} \\ &= \begin{pmatrix} -\alpha I & 0 \\ \\ (\alpha - \beta)I - \delta_1 & -\beta S - (\delta_1 + \ \delta_2) \end{pmatrix} \end{aligned} \tag{8.41}$$

Recall that the eigenvalues of A are the roots of the characteristic polynomial

$$c_A(z) = \det(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - A) = (z + \alpha I)(z + \beta S + \delta_1 + \delta_2) \tag{8.42}$$

Thus the eigenvalues of A are

$$-\alpha I \quad \text{and} - \beta S - \delta_1 - \delta_2.$$

These are both real numbers, and they are both negative if and only if $I \neq 0$. From (7.32) we get $I = 0$ if and only if $\delta_3 = \alpha$. Since we have assumed $\delta_3 \leq \alpha$ we have this conclusion:

Theorem 8.2. *When* $\delta_3 \leq \alpha$, *the equilibrium*

$$x_0 = \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} \frac{\delta_3}{\alpha} \\ \\ \frac{\delta_1(1-\frac{\delta_3}{\alpha})}{\delta_1 + \frac{\beta}{\alpha}\delta_3 + \delta_2} \end{pmatrix}$$

is linearly stable if and only if $\delta_3 < \alpha$.

Note that if $\delta_3 = \alpha$ then we are again at the equilibrium in which $S = 1$ and $I = 0$, but this not linearly stable.

References

[1] Iannelli, M. The Mathematical Modeling of Epidemics 2005 Summer School on Mathematical Models in Life Sciences: Theory and Simulation.