NOTES ON EPIDEMIC MODELS

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1. The model

The goal of these notes is to set up a fairly general epidemic model which can be applied to a number of different situations.

The model has three different populations:

1. The "susceptible" population, of size $S(t)$ at time t.
2. The "infected" population, of size $I(t)$ at time t.
3. The "removed" population, of size $R(t)$ at time t.

Here are two examples.

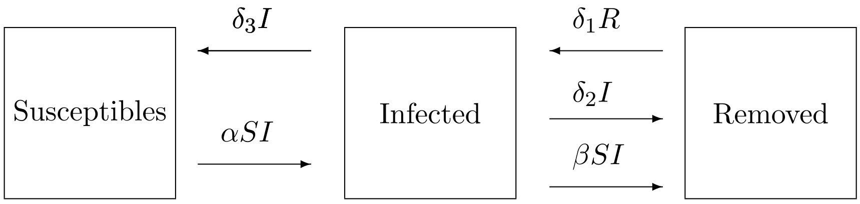
Example 1.1. In the covid-19 pandemic, the "susceptible" population consists of people who are not now infected but who could become so. The "infected" population are those who are currently infected. The "removed" population could be the people who are currently immune to infection. Note that some of those currently immune might lose their immunity over time.

Example 1.2. The model will apply also to zombie epidemics. For example, suppose we say that someone has become zombified if they are unable to consider evidence which contradicts their prior beliefs. The "susceptible" population then consists of those people who have not become zombified, but who could become so at some point. This group is willing to consider new information, even if it contradicts their prior beliefs. The "infected" population consists of people who are zombified in the above sense. The "removed" population consists of those people who have removed themselves from interactions with other people over the controversy in question.

2. Dynamics of the model

We assume that there are positive constants $\alpha, \beta, \delta_1, \delta_2, \delta_3$ for which the following is true:

1. The susceptible and infected populations interact at a rate proportional to the product $S(t)I(t)$ of their numbers. The outcome of such an interaction can be no change, the infected could become removed, or the susceptible person could become infected. The latter conversions occur at rates $\beta S(t)I(t)$ and $\alpha S(t)I(t)$, respectively.
2. A removed person may be return to the infected population. The rate at this this occurs is proportional to $R(t)$ and it given by $\delta_1 R(t)$.
3. Members of the infected population may return to being susceptible at a rate $\delta_3 I(t)$ or become removed at rate $\delta_2 I(t)$.



Let's consider how to interpret the constants in this model in the two examples of the previous section.

2.1. **The covid-19 case.** It is reasonable to suppose that the rate of interactions between susceptible and infected people is on average proportional to the product of the sizes of their population. This leads to the term αSI in the rate at which susceptibles become infected.

At present, the interaction of a susceptible person with an infected person does have a certain probability of leading to the infected person recovering and becoming part of the "removed" group. For example, some fraction of the susceptible population consists of health care workers. We might assume that there rate at which interactions occur between infected people and health care workers is proportional to the product of the infected and susceptible populations. We assume in the model that such interactions lead with a certain rate per interaction to the infected individuals becoming removed (rather than immediately susceptible again). This leads to the term βSI of infected individuals becoming removed.

Infected individuals can spontaneously recover either to become susceptible or removed, at possibly different rates δ_3 and δ_2. Finally, we are supposing that removed individuals, who are currently not susceptible, may return to being infected at some rate δ_1.

2.2. **The zombie epidemic case.** We are supposing that the number of interactions per day between non-zombified and zombified individuals is proportional to the product of the sizes of their populations. Such interactions either result in the zombification of the non-zombified person, or the transfer of the zombified person to the "removed" population. In the classic zombie movie scenario, the latter occurs when a zombie is physically subdued and goes dormant, often only to return to a zombified state a short time later. The latter event results in the decay of members of the removed population back to the infected, zombie population.

The rate at which infected, zombified individuals decay back into being susceptible or removed are δ_3 and δ_1. In a typical zombie movie, one does not observe zombies spontaneously getting better and becoming susceptible but unzombified. However, politically zombified people interact less well with reality than those people who are susceptible to new information. Bad outcomes when coming into contact with reality may in fact lead to a small proportion of people with zombified views becoming willing to accept information that contradicts their prior beliefs. This phenomenon has certainly occurred during the covid-19 pandemic.

3. Differential equations

The autonomous differential equations which result from the model of the previous section are:

$$\frac{dS}{dt} = -\alpha SI + \delta_3 I \tag{3.1}$$

$$\frac{dI}{dt} = (\alpha - \beta)SI - (\delta_3 + \delta_2)I + \delta_1 R \tag{3.2}$$

$$\frac{dR}{dt} = \beta SI + \delta_2 I - \delta_1 R \tag{3.3}$$

Note that

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0.$$

So $S + I + R$ is constant, and we chooose to normalize this constant by

$$S(t) + I(t) + R(t) = 1 \quad \text{for all} \quad t \tag{3.4}$$

Thus S, I and R now represent the fractions of the total population represented by members of each group. One can thus consider SI to be the probability that a randomly chosen ordered pair of people will consist of first a susceptible and second an infected (or zombified) person.

Notice that (3.4) says that there are really only to independent functions in this system, rather than three. We can make this explicit in the following way. Substitute

$$R = 1 - (I + S)$$

into the above differential equations to have the two variable autonomous system of differential equations given by

$$\frac{dS}{dt} = -\alpha SI + \delta_3 I = G_1(S, I) \tag{3.5}$$

$$\begin{aligned} \frac{dI}{dt} &= (\alpha - \beta)SI - (\delta_3 + \delta_2)I + \delta_1(1 - (I + S)) \\ &= (\alpha - \beta)SI - (\delta_1 + \delta_2 + \delta_3)I - \delta_1 S + \delta_1 \\ &= G_2(S, I) \end{aligned} \tag{3.6}$$

If we let $f : \mathbb{R} \to \mathbb{R}^2$ be

$$f(t) = \begin{pmatrix} S(t) \\ I(t) \end{pmatrix}$$

and

$$G(S, I) = \begin{pmatrix} G_1(S, I) \\ G_2(S, I) \end{pmatrix}$$

then the system of differential equations is

$$\frac{df}{dt}(t) = G(f(t)). \tag{3.7}$$

4. Lecture 1 of "The Mathematical Modeling of Epidemics" by M. Iannelli

The model discussed in equation (4) of [1] amounts to these equations:

$$\frac{dS}{dt} = -\alpha SI \tag{4.8}$$

$$\frac{dI}{dt} = \alpha SI - \delta_2 I \tag{4.9}$$

$$\frac{dR}{dt} = \delta_2 I \tag{4.10}$$

For further interpretation of the constant α, see equations (1) and (2) of [1]. Recall that we are normalizing the constant $N = S(t) + I(t) + R(t)$ to be 1. This is not done in [1], so one has to take this into account when translating between these notes and [1]. We are focusing on the fraction $S(t)$ of the total population which is susceptible, for example, rather than on the absolute number of susceptibles.

The above specialization of the equations of the previous section amounts to the following additional hypotheses.

A. Interactions between susceptible individuals and infected individuals don't lead to infected individuals becoming removed, i.e. $\beta = 0$ in the previous section.
B. Infected individuals never spontaneously become susceptible again.
C. Removed individuals never become infected again.

This specialization of the prior model does not cover the typical zombie movie scenario very well, in which zombies who have been forced into a removed state spontaneously return to being zombies.

Anyone who has seen a typical zombie movie knows that usually, zombies triumph. However, we will see this is not the case in the above specialization. We now recap the derivation of the following result given in [1]:

Theorem 4.1. *Suppose $\alpha > 0$ and $\delta_2 > 0$. Let $S(t)$, $I(t)$ and $R(t)$ satisfy the differential equations (4.8), (4.9) and (4.10), and suppose*

$$1 = S(0) + I(0) + R(0)$$

with $S(0) > 0$, $I(0) > 0$ and $R(0) \geq 0$. Then $S(t)$ decreases monotonically as $t \to \infty$ to a positive limit S_∞ which satisfies the equation

$$S_\infty - \frac{\delta_2}{\alpha} \ln S_\infty = S(0) + I(0) - \frac{\delta_2}{\alpha} \ln S(0). \tag{4.11}$$

Furthermore, $\lim_{t\to\infty} I(t) = 0$ and $I(t) > 0$ for all $t \geq 0$.

Proof. From (4.8) and (4.9) we get

$$\frac{dS}{dt} \leq 0 \quad \text{and} \quad \frac{d(S+I)}{dt} = -\delta_2 I \leq 0 \tag{4.12}$$

Since $0 \leq S(t) \leq 1$, we conclude that $S(t)$ has non-increasing limit $S_\infty = \lim_{t\to\infty} S(t)$ as $t \to \infty$. We get from (4.10) that

$$0 \leq \delta_2 \int_0^t I(s)ds = R(t) - R(0) = (1-S(t)-I(t)) - (1-S(0)-I(0)) = S(0)-S(t)+I(0)-I(t) \leq 1.$$

Now $I(s) \geq 0$ for all s means $\delta_2 \int_0^t I(s)ds$ cannot decrease as t increases, so since we have shown this integral is bounded, we see that $\delta \int_0^\infty I(s)ds$ converges. So

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} S(0) - S(t) + I(0) - \int_0^t I(s)ds = S(0) - S_\infty + I(0) - \int_0^\infty I(s)ds$$

is well defined. However, this forces

$$\lim_{t\to\infty} I(t) = 0$$

because the integral $\int_0^\infty I(s)ds$ converges.

We now use the fact that (4.8) and (4.9) imply

$$\frac{d}{dt}(S + I - \frac{\delta_2}{\alpha}\ln S) = -\alpha SI + \alpha SI - \delta_2 I - \frac{\delta_2}{\alpha}\frac{dS/dt}{S} = 0.$$

So $S + I - \frac{\delta_2}{\alpha}\ln S$ is constant, and this gives (4.11).

Finally we need to check that S_∞ are $I(t)$ are positive. From (4.8) we get

$$\int_0^\infty \frac{d}{ds}\ln S ds = \int_0^\infty -\alpha I(s)ds$$

converges. This implies $\ln S(t)$ has a finite limit as $t \to \infty$, so S_∞ must be positive. From (4.9) we get

$$\frac{d}{dt}\ln I = \alpha S - \delta_2$$

So

$$\ln I(t) - \ln I(0) = \int_0^t \frac{d}{ds}\ln I ds = \int_0^t (\alpha S(s) - \delta_2)ds$$

is finite for all $t \geq 0$ and we conclude that $I(t)$ can never be 0. Now (4.8) shows $S(t)$ is monotonically decreasing, rather than just being non-increasing. □

Theorem 4.2. *If $S(0) < \delta_2/\alpha$, then $I'(t) < 0$ for all $t > 0$. Suppose now that $S(0) > \delta_2/\alpha$. There will be a unique time t^* such that $S(t^*) = \delta_2/\alpha$. One has in this case that $I'(t) > 0$ for $t < t^*$ and $I'(t) < 0$ for $t > t^*$.*

Proof. From (4.9) we get

$$\frac{dI}{dt} = (\alpha S - \delta_2)I \tag{4.13}$$

Since we showed $I(t) > 0$ of for all t and $S(t)$ is non-increasing with t, we see that $\frac{dI}{dt} < 0$ for all t if $\alpha S(0) < \delta_2$. Suppose now that $S(0) > \delta_2/\alpha$. Since we showed $\lim_{t\to\infty} I(t) = 0$, $\frac{dI}{dt}$ has to be negative for some value of t. Since we showed $S(t)$ is monotonically decreasing, we conclude from (4.13) that there will be unique t^* such that $\alpha S(t^*) - \delta_2 = 0$. The final statement in the Theorem is now clear from (4.13) and $I(t) > 0$ for all t.

□

Remark 4.3. The significance of Theorem 4.2 is that if $S(0) < \delta_2/\alpha$, there is a steady decline in the infected population, while if $S(0) > \delta_2/\alpha$ this population increases for awhile and then decreases. The peak happens when $t = t^*$, which is the time of the height of the resulting epidemic. This accounts for the difference between the steadily declining infected population curve and the peaked infected population curve in Figure 8 of [1].

References

[1] Iannelli, M. The Mathematical Modeling of Epidemics 2005 Summer School on Mathematical Models in Life Sciences: Theory and Simulation.