Joint Density function

$$f = f_{X_1, \dots, X_n} : \mathbb{R}^n \to \mathbb{R}$$

of random variables

$$X_1, \dots, X_n : S \to \mathbb{R}$$

1) S discrete

$r_1, \dots, r_n \in \mathbb{R}$

$$f(r_1, \dots, r_n) = \text{Prob}(X_1 = r_1, \dots, X_n = r_n)$$

2) Continuous case

$A_1, \dots, A_n =$ intervals $\subseteq \mathbb{R}$

$$\text{Prob}(X_1 \in A_1, \dots, X_n \in A_n)$$

$$= \int_{A_1} \int_{A_2} \dots \int_{A_n} f(r_1, r_2, \dots, r_n)\, dr_1 \dots dr_n$$

Fact: $X_1, \dots, X_n$ are independent if

$$f_{X_1,\dots,X_n}(r_1,\dots,r_n) =$$

$$f_{X_1}(r_1) \cdot f_{X_2}(r_2) \cdots f_{X_n}(r_n)$$

EX: $S = \{(A_1,\dots,A_n) ; A_i = 0 \text{ or } 1\}$

$$X_i : S \to \{0,1\}$$

$$(A_1 \dots, A_n) \to A_i$$

$$f_{X_i}(1) = \text{Prob}(X_i = 1) = p_i$$

$$f_{X_i}(0) = 1 - p_i$$

Assume all $p_i = p$ and $X_1, \dots, X_n$ independent

$$f_{X_1,\dots,X_n}(A_1,\dots,A_n) =$$

$$p^{\#\{i ; A_i = 1\}} \cdot (1-p)^{\#\{j ; A_j = 0\}}$$

$$X = X_1 + \dots + X_n : S \to \mathbb{Z}_{\geq 0}$$

$$X((a_1, \dots, a_n)) = \#\{i \,;\, a_i = 1\}$$

$$\cancel{f_X(j)=}$$

$$f_X(j) = \Pr(X = j) = \binom{n}{j} p^j (1-p)^{n-j}$$

Binomial density function
for parameters n and p

Variance of $Y : S \to \mathbb{R}$

Recall $E(Y) =$ expectation

$$= \begin{cases} \sum_r r \cdot f_Y(r) \\ \int_{r=-\infty}^{\infty} r f_Y(r)\,dr \end{cases}$$

$$\mathrm{Var}(Y) = E\left((Y - E(Y))^2\right)$$

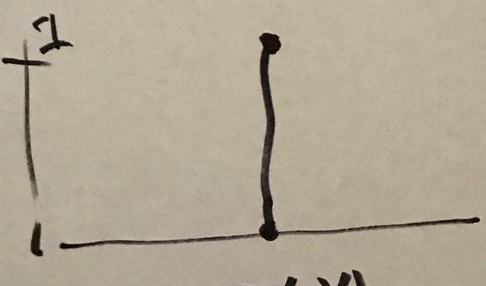
$$\sigma(Y) = \text{standard deviation}$$
$$= \sqrt{\mathrm{Var}(Y)}$$

Heuristic: $\mathrm{Var}(Y)$ and $\sigma(Y)$ measure the spread of values of Y around the ~~m~~ mean

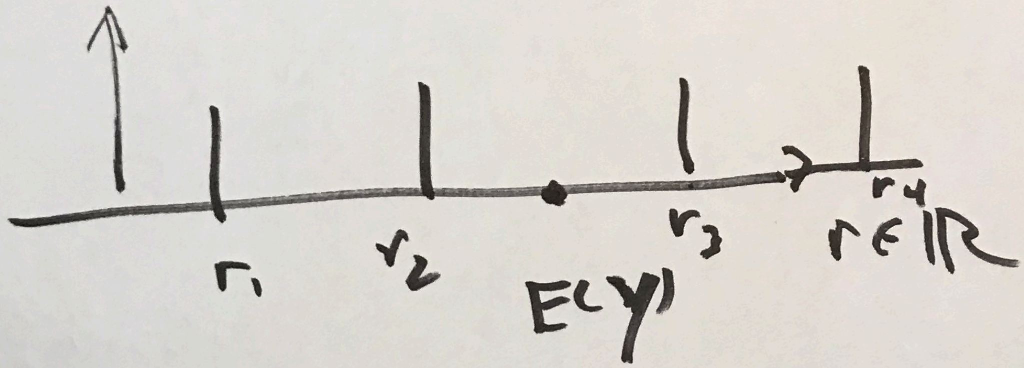
E.G: Suppose $\mathrm{Prob}(Y = E(Y)) = \underline{1}$.

$$\mathrm{Prob}\left((Y - E(Y))^2 = 0\right) = 1$$

$$E\left((Y - E(Y))^2\right) = 0$$



Example $z = f_Y(r)$



$$\mathrm{var}(Y) = E\big((Y - E(Y))^2\big)$$

large

Fact: Let $h : \mathbb{R} \to \mathbb{R}$

$$h(Y) : S \to \mathbb{R}$$
$$s \to h(Y(s))$$

$E(h(Y))$ is

$$\begin{cases} \sum_r h(r) f_Y(r) & \text{discrete case} \\ \int h(r) f_Y(r)\, dr & \text{cont. case} \end{cases}$$

Discrete case

$$E(h(Y)) = \sum_{r'} r' \cdot \text{Prob}(h(Y) = r')$$

$$= \sum_{r'} \sum_{\substack{r \text{ with} \\ h(r) = r'}} r' \cdot \text{Prob}(Y = r)$$

$$= \sum_{r} h(r) \, \text{Prob}(Y = r)$$

$$= \sum_{r} h(r) \, f_Y(r)$$

Application ~~to~~ To $\text{Var}(Y) = E\left((Y - E(Y))^2\right)$

Let $h(r) = (r - E(Y))^2$

So $h(Y) = (Y - E(Y))^2$

Example: $Y : S \to \{0,1\}$

$$f_Y(1) = p\,,\quad f_Y(0) = 1-p = q$$

$$E(Y) = 0 \cdot f_Y(0) + 1 \cdot f_Y(1) = p$$

$$\mathrm{Var}(Y) = E((Y-p)^2)$$

$$h : \mathbb{R} \to \mathbb{R} \qquad h(r) = (r-p)^2$$

$$\mathrm{Var}(Y) = \sum_{r=0,1} h(r) f_Y(r)$$

$$= p^2 q + (1-p)^2 p$$

$$= h(0) f_Y(0) + h(1) f_Y(1)$$

$$= p^2 q + q^2 p = (p+q) pq = pq$$

$$\sigma(Y) = \sqrt{pq} \qquad q = 1-p$$

<u>Theorem</u>: Suppose $X_1, \dots, X_n : S \to \mathbb{R}$ are <u>independent</u>. Then

1) $Var(X_1 + \dots + X_n) = Var(X_1) + \dots + Var(X_n)$

2) $E(X_1 \cdot X_2 \cdots X_n) = E(X_1) \cdots E(X_n)$

<u>Note</u>: Even if X_i are dependent, always have $E(X_1 + \dots + X_n) = E(X_1) + \dots + E(X_n)$

<u>Proof when $n=2$</u>

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

$$E\left(\left(X_1 + X_2 - (E(X_1) + E(X_2))\right)^2\right) = Var(X_1 + X_2)$$

$$\parallel$$

$$E\left(\left(X_1 - E(X_1) + X_2 - E(X_2)\right)^2\right)$$

$$= E\left((X_1 - E(X_1))^2\right) + E\left((X_2 - E(X_2))^2\right) + T$$

$$T = E\big(2 \cdot (X_1 - E(X_1)) \cdot (X_2 - E(X_2))\big)$$

Want: $T = 0$

Use joint density

$$f = f_{X_1, X_2} : \mathbb{R}^2 \to \mathbb{R}$$

Discrete case

$$T = \sum_{r_1, r_2} 2(r_1 - E(X_1)) \cdot (r_2 - E(X_2))\, f(r_1, r_2)$$

$$\overset{\text{independence}}{=} \sum_{r_1, r_2} 2(r_1 - E(X_1))(r_2 - E(X_2))\, f_{X_1}(r_1) \cdot f_{X_2}(r_2)$$

$$= 2\Big(\sum_{r_1} \cancel{2}(r_1 - E(X_1))\, f_{X_1}(r_1)\Big) \cdot \sum_{r_2} \cancel{2}(r_2 - E(X_2))\, f_{X_2}(r_2)$$

$$= 2 \cdot 0 \cdot 0$$

Corollary: Suppose

$$X_1, \dots, X_n : S \to \mathbb{R}$$

are independent and each one has the same density function

$$f_{X_i} = h : \mathbb{R} \to \mathbb{R}$$

Then $X = X_1 + \dots + X_n$ has

$$E(X) = n\, E(X_1)$$

$$Var(X) = n\, Var(X_1)$$

$$\sigma(X) = \sqrt{Var(X)} = \sqrt{n}\, \sigma(X_1)$$

eg. Each X_i is Bernoulli trial param. p

$$E(X) = np$$

$$\sigma(X) = \sqrt{n}\sqrt{p(1-p)}$$

Central Limit Theorem

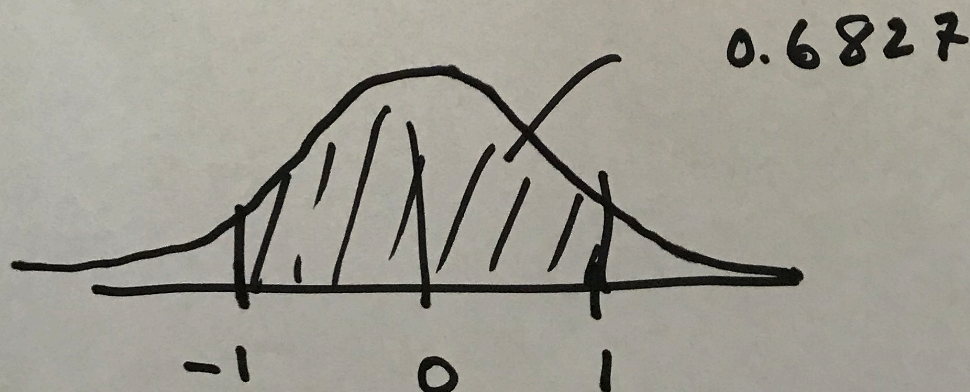
$\{X_i\}_{i=1}^{n}$ independant, same density function

$$E(X_i) = \mu \ , \ Var(X_i) = \sigma^2$$

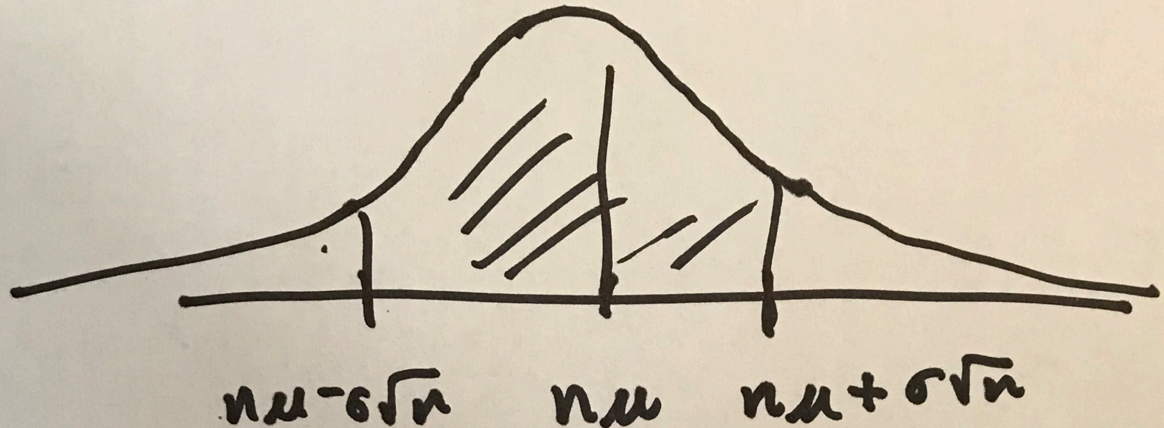
$$Y = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

<u>Theorem</u>: As $n \to \infty$, the density function of Y converges to the "normal" density

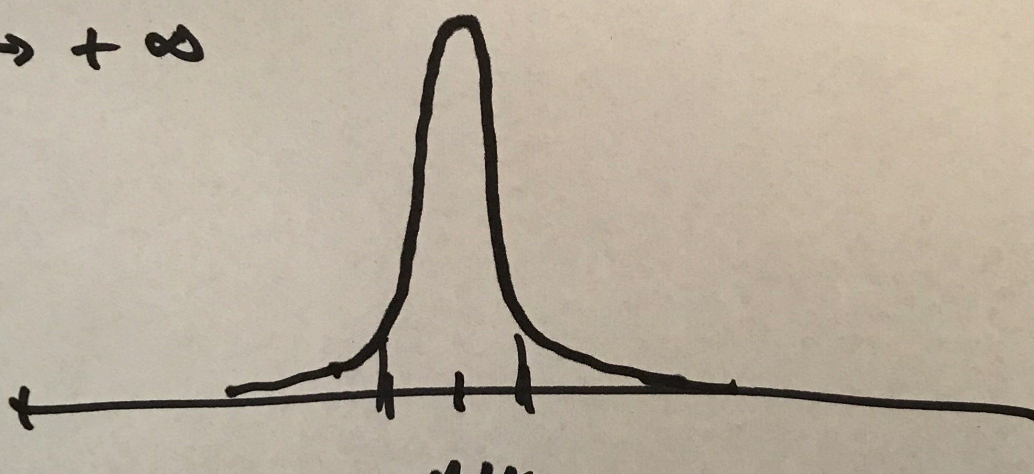
$$f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$



$$X_1 + \cdots + X_n = n\mu + \sigma\sqrt{n}\, Y$$



$n \to +\infty$



X_i = Bernoulli trial prob p

$\mu = p \qquad \sigma = \sqrt{p(1-p)}$

$$\text{Prob}(X_1 + \cdots + X_n = j) = \binom{n}{j} p^j (1-p)^{n-j}$$

Example: In a large population, the odds a person will have covid-19 are p. Pick a sample size of n. What is probability density associated with the number of active cases in this sample?

$S =$ All choices of samples of size n

$X_i : S \to \mathbb{R}$, $X_i = 1$ if ith member of sample has covid-19, 0 else

$X = X_1 + \cdots + X_n : S \to \mathbb{R}$, $X = $ # of cases in sample

$f_X =$ binomial, $f_X(j) = \binom{n}{j} p^j (1-p)^{n-j}$

Central Limit Theorem For n large

$$\text{Prob}\left(np - \sigma\sqrt{n} \le X \le np + \sigma\sqrt{n}\right)$$

$$\simeq \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \simeq 0.6827\ldots$$

Ex: $n = 1000$

$p = 0.01$

$np = 10$ cases expected in a sample of 1000 people

$$\sigma = \sqrt{p(1-p)} = \sqrt{0.01 \cdot 0.99} \approx .1$$

$$\sqrt{n}\,\sigma \approx 3.16$$

Central Limit

$$\text{Pr}\left(10-3 \le X \le 10+3\right) \approx .6827$$

$$= \sum_{j=7}^{13} \binom{1000}{j} p^j (1-p)^{1000-j} = 0.73$$