Poisson Process

Count occurrences in some time interval. Subdivide interval into n successive subintervals of same length. Suppose as $n \to +\infty$, there's at most one occurrence in each subinterval, with prob. $p = \frac{\lambda}{n}$, and occurrences in different intervals are independent.

X = total no. of occurences

$$\text{Prob}(X = j) = \binom{n}{j} p^j (1-p)^{n-j}$$

$j = 0, 1, \ldots$

Let $n \to \infty$, $p = \frac{\lambda}{n}$, λ = fixed.

$f(j) = \mathrm{Prob}(X = j)$

$$\frac{f(j+1)}{f(j)} = \frac{\binom{n}{j+1} p^{j+1} (1-p)^{n-(j+1)}}{\binom{n}{j} p^j (1-p)^{n-j}}$$

$$= \frac{(n-j)\, p}{(j+1)(1-p)}$$

If $n >> j$

$p = \frac{\lambda}{n} \sim 0$

$$\cong \frac{n \cdot \lambda/n}{(j+1) \quad 1}$$

$$= \frac{\lambda}{j+1}.$$

$$f(j) = \frac{\lambda}{j} f(j-1) = \frac{\lambda \cdot \lambda}{j(j-1)} f(j-2)$$

$$= \cdots = \frac{\lambda^j}{j!} f(0)$$

$$1 = \text{Prob}(X \geq 0) = \sum_{j=0}^{\infty} f(j)$$

$$= \sum_{j=0}^{\infty} \frac{f(0)\lambda^j}{j!} = f(0) \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= f(0)\, e^{\lambda}. \quad \text{So } f(0) = e^{-\lambda}$$

<u>Def</u>: X has Poisson distribution with rate λ if

$$f_X(j) = \text{Prob}(X = j)$$

$$= \begin{cases} e^{-\lambda} \dfrac{\lambda^j}{j!} & \text{if } j = 0, 1, \ldots \\ 0 & \text{else} \end{cases}$$

Fact: $E(X) = \sum_{j=0}^{\infty} f_X(j) \cdot j$

$$= \sum_{j=0}^{\infty} e^{-\lambda} \frac{\lambda^j}{j!} j = e^{-\lambda} \sum_{j=1}^{\infty} \frac{\lambda \cdot \lambda^{j-1}}{(j-1)!}$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda$$

Ex: λ = rate of new infections caused by a covid-19 patient, without social distancing

$\lambda = 2.5$

$f_X(0) = \text{Prob}\begin{pmatrix}\text{no infections} \\ \text{caused}\end{pmatrix}$

$$= \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda} = 0.082$$

Exponential Distribution

Problem: Suppose a person with covid-19 recovers in 10 days on average. What is the probability they will require at least 10 days?

$\tilde{\lambda}$ = recovery rate per second

Rate of recovery over time $[0, t]$

$$\lambda = \tilde{\lambda} t$$

Prob. of no recovery in $[0, t]$

$$= \text{Prob}\left(\begin{array}{c}\text{Poisson Variable for } \lambda \\ \text{takes value } 0\end{array}\right)$$

$$= \frac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-\lambda}$$

$Y =$ recovery time
(time of first event)

$$\text{Prob}(Y \le t) = 1 - \text{Prob}\begin{pmatrix}\text{no recovery} \\ \text{in } [0,t]\end{pmatrix}$$

$$= 1 - e^{-\lambda}$$

$$= 1 - e^{-\tilde{\lambda} t}$$

$\text{Prob}(Y \le t) = F_Y(t) =$ distribution function of Y

$$f_Y(t) = \frac{d}{dt} F_Y(t) = \tilde{\lambda}\, e^{-\tilde{\lambda} t}$$

<u>Def</u>. Y is exponentially distributed with rate $\tilde{\lambda}$.

Calculate $E(Y) = \int_{-\infty}^{\infty} t f_Y(t)\, dt$

$$= \int_0^{\infty} t \tilde{\lambda} e^{-\tilde{\lambda} t} dt$$

$$= \frac{1}{\tilde{\lambda}}$$

integrate by parts

Make Sense: Expected time of recovery $= \frac{1}{(\text{rate})}$

Time to recovery from covid-19 $= Y$

$E(Y) = 10$ days $= \frac{1}{\tilde{\lambda}}$

$$\text{Prob}(Y \geq 10) = \int_{1/\tilde{\lambda}}^{\infty} \tilde{\lambda} e^{-\tilde{\lambda} t} dt = e^{-1}$$

$$= 0.367\ldots$$