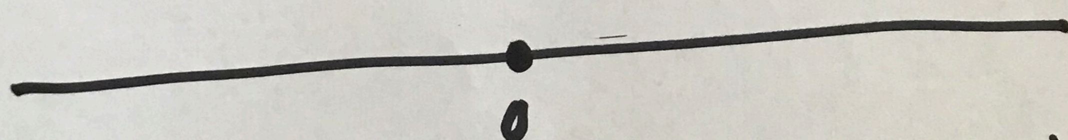
Random Walks

Scenario: A particle flips a fair coin at $t = 0, 1, 2, \ldots$ to decide whether to move $+1$ to right or -1 to left on the real line



What's the probability density function of its position at time $t = n$?

What's the expected distance travelled?

Sample Space

$$S = \{ s = (s_1, \dots, s_n) \; ; \; s_i \in \{-1, 1\} \}$$

$$X_i : S \longrightarrow \{-1, 1\}$$

$$s \longrightarrow s_i = X_i(s)$$

$X =$ position at step n $= X_1 + \dots + X_n$

$E(X_i) = 0$ since $\text{Prob}(X_i = 1) = 1/2$

$\text{Prob}(X_i = -1) = 1/2$

$$\sigma_{X_i} = \sqrt{\text{Var}(X_i)} = \sqrt{1/2 \, (1 - 1/2)} = 1/2$$

$$E(X) = \sum_{i=1}^{n} E(X_i) = 0$$

$$\text{Var}(X) = \sum_i \text{Var}(X_i) = n \cdot 1/4$$

$$\sigma_X = \sqrt{n} \; 1/2 = \sqrt{n} \; \sigma_{X_1}$$

Central Limit Theorem

The density function of

$$Y_n = \frac{(X_1 + \cdots + X_n) - E(X_1 + \cdots + X_n)}{\sqrt{n}\, \sigma_{X_1}}$$

$$= \frac{X_1 + \cdot + X_n}{\sqrt{n}\ 1/2} = \frac{2(X_1 + \cdots + X_n)}{\sqrt{n}}$$

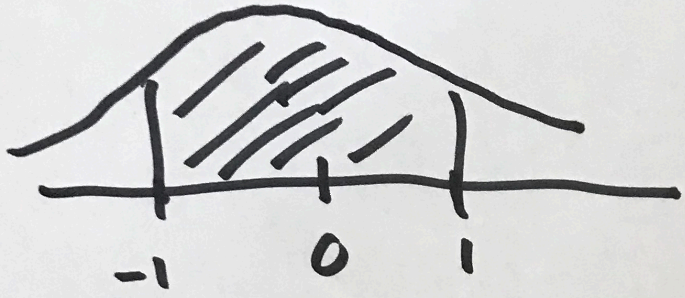
converges as $n \to +\infty$ to normal density $f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$

For $\ell \geq 0$

$$\text{Prob}\left(-\frac{\sqrt{n}\,\ell}{2} \leq X_1 + \cdots + X_n \leq \frac{\sqrt{n}\,\ell}{2}\right)$$

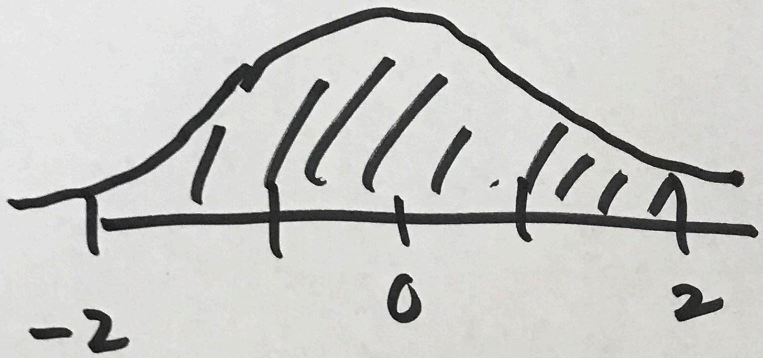
$$= \text{Prob}(-\ell \leq Y_n \leq \ell) \approx \int_{-\ell}^{\ell} f(t)\,dt$$

$\ell = 1$



$P \approx 0.68$

$\ell = 2$



$P \approx 0.95$

What is

$E(|X_1 + \cdots + X_n|) =$ expected distance from starting point.

$$\| \quad E\left(\frac{1}{2}\sqrt{n}\,|Y_n|\right) = \frac{\sqrt{n}}{2} E(|Y_n|)$$

$$f_{Y_n}(t) \approx f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

As $n \to \infty$, have

$$E(|Y_n|) = E(|Z|) \text{ when } f_Z = f(t)$$

$$= \int_{t=-\infty}^{\infty} |t|\, f_Z(t)\,dt$$

$$= \int_{t=-\infty}^{\infty} |t| \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$$

$$= 2\int_{t=0}^{\infty} \frac{1}{\sqrt{2\pi}}\, t\, e^{-t^2/2}\,dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_{t=0}^{\infty} e^{-t^2/2}\, d\left(t^2/2\right)$$

$$= \frac{2}{\sqrt{2\pi}} \int_{v=0}^{\infty} e^{-v}\,dv = \frac{2}{\sqrt{2\pi}}$$

$$E(|X_1 + \cdots + X_n|) = \frac{1}{2}\sqrt{n} \cdot \frac{2}{\sqrt{2\pi}} = \frac{\sqrt{n}}{\sqrt{2\pi}}$$

expected dist. from 0.

Virus Example

Suppose

$\text{Prob}(|X_1 + \dots + X_n| \le d) = 0.999$

means distance d is "safe".

n = proportional to time.

If we replace n by $N = 2n$

what should $d \to D$ = new distance

to have same safety?

$$\text{Prob}(|X_1 + \dots + X_n| \le d) = \text{Prob}\left(|Y_n| \le \frac{2d}{\sqrt{n}}\right)$$

$$\cong \text{Prob}\left(|Z| \le \frac{2d}{\sqrt{n}}\right)$$

$$\text{Prob}(|X_1 + \dots + X_{2n}| \le D) = \text{Prob}\left(|Z| \le \frac{2D}{\sqrt{2n}}\right)$$

$$\frac{2d}{\sqrt{n}} = \frac{2D}{\sqrt{2n}} \Rightarrow \frac{D}{d} = \sqrt{2}$$

$d = 6$ ft $\quad D = \sqrt{2} \cdot 6 = 8.48$ ft