Probability + Counting

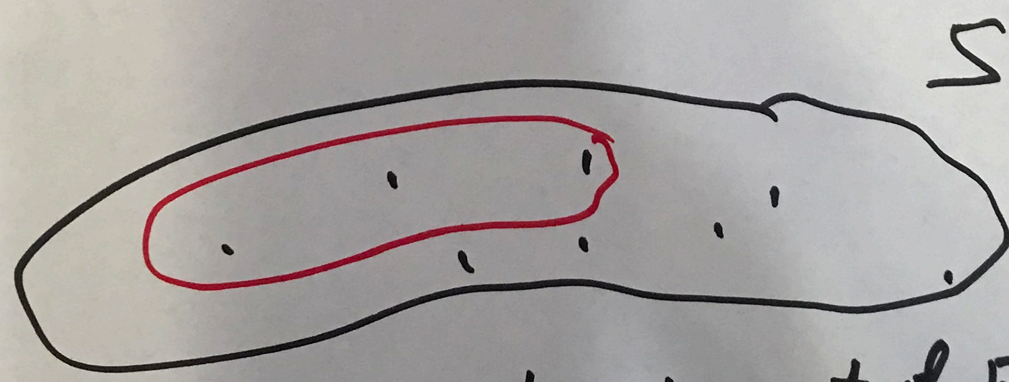
$S = \{s_1, \cdots, s_m\}$ = finite sample space

$\mathcal{B} = 2^S$ = all possible subsets of S

Say each $\{s_i\}$ has $P(\{s_i\}) = \frac{1}{m}$

For all $E \subseteq S$,

$$P(E) = \frac{\# E}{m}$$



Ways to count elements of E.

Suppose S is the set of all ordered (or unordered) sequences $t_1, \ldots, t_n$ of n elements of some other set T.

<u>Example</u>: A lottery ticket consists of 6 unordered numbers chosen from $\{1, \ldots, 51\}$.

S = set of all sets of 6 numbers chosen from $\{1, \ldots, 51\}$.

$E = \{A_1\}$ A_1 = one ticket

What is $\underline{P}(E)$?

Permutations: The number of **ordered** sequences $t_1, \ldots, t_n$ of $n \leq q = \# T$ elements from T is

$$q(q-1)\cdots(q-n+1) = q!/(q-n)!$$

Combinations: The number of **unordered** subsets $\{t_1, \ldots, t_n\}$ of n elements from T is

$$\binom{q}{n} = \frac{q(q-1)\cdots(q-n+1)}{n!} = \frac{q!}{n!(q-n)!}$$

E.G. $n = 6$, $q = 51$, $E =$ one ticket

$$\underline{P}(E) = \frac{1}{\binom{51}{6}} = \frac{1}{18{,}009{,}640}$$

Multinomial Theorem: Suppose $t \geq 1$ and $n = n_1 + \cdots + n_t$ with $n_1, \ldots, n_t \geq 0$.

The number c of ways of picking disjoint unordered subsets $T_1, \ldots, T_t$ of $\{1, \ldots, n\}$ so $\# T_i = n_i$

is
$$c = \frac{n!}{n_1!\, n_2! \cdots n_t!}$$

e.g. $t = 2$, $n = n_1 + n_2$, $n_2 = n - n_1$

$$c = \frac{n!}{n_1!\,(n-n_1)!} = \binom{n}{n_1}$$

__Why?__ For each choice of $T_1, \ldots, T_t$ we have $n_1!\, n_2! \ldots n_t!$ ways of ordering the elements of $T_1, \ldots, T_t$. Get $c \cdot n_1! \ldots n_t!$ ways of picking __ordered__ disjoint sets $\tilde{T}_1, \ldots, \tilde{T}_t$ with $\# \tilde{T}_i = n_i$. List these. Get the $n!$ ways of listing the n elements of T.

$$c \cdot n_1! \ldots n_t! = n!$$

Example: How many English words can be made by reordering the letters in ddd a e ?

$n = 5$ positions in which to put the letters

$n_1 = 3$ d's

$n_2 = 1$ a's

$n_3 = 1$ e's

Need: Choose $T_1, T_2, T_3 \subseteq \{1, .., 5\}$

T_1 = positions of the d's

T_2 = position of a

T_3 = position of e

Multinomial theorem:

There are

$$\frac{n!}{n_1!\,n_2!\,n_3!} = \frac{5!}{3!\cdot 1!\,1!} = 5\cdot 4 = 20$$

ways to pick T_1, T_2, T_3.

dddae
dddea
ddeda
ddade
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