Conditional Probability + Bayes Theorem

Idea: Find the odds that some event A occurs given that another event C is known to occur.

Def: Suppose $P(C) \neq 0$

Then

$$P(A|C) = \frac{P(A \cap C)}{P(C)}$$

Recall: A, C = subsets of a sample space S

Example

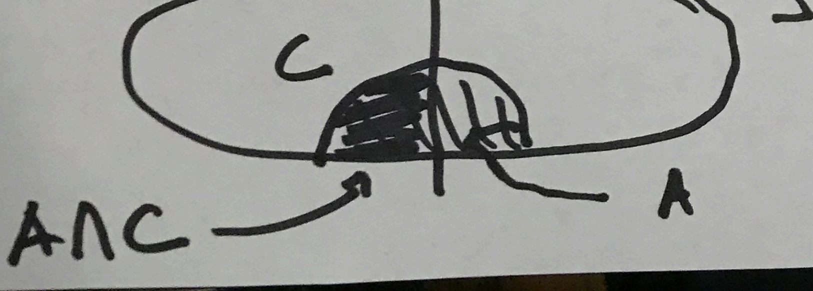
S = finite sample space and all elements of S are equally likely.

$$\underline{P}(A) = \#A / \#S$$

$$\underline{P}(A|C) = \frac{\underline{P}(A \cap C)}{P(C)}$$

$$= \frac{\#(A \cap C) / \#S}{\#C / \#S}$$

$$= \frac{\#(A \cap C)}{\#C}$$



Common Confusion:

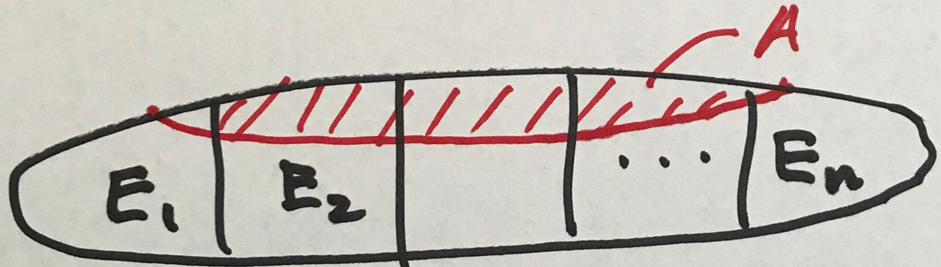
Claims that A and C "go together" are often ambiguous. Does this mean $\underline{P}(A|C)$ is large, or that $\underline{P}(C|A)$ is large.

Bayes Theorem

S = Sample space

$= E_1 \cup E_2 \cdots \cup E_n$

with $E_1, \ldots, E_n$ disjoint events



A = another event

How can we find

$\mathbb{P}(E_i \mid A)$ = probability of E_i given that A has occurred?

Bayes Theorem

$$\underline{P}(E_i|A) = \frac{\underline{P}(E_i \cap A)}{\underline{P}(A)}$$

$$= \frac{\underline{P}(A|E_i)\cdot \underline{P}(E_i)}{\underline{P}(A|E_1)\,\underline{P}(E_1) + \cdots + \underline{P}(A|E_n)\,\underline{P}(E_n)}$$

Proof: $E_1, \ldots, E_n$ = disjoint partition of S

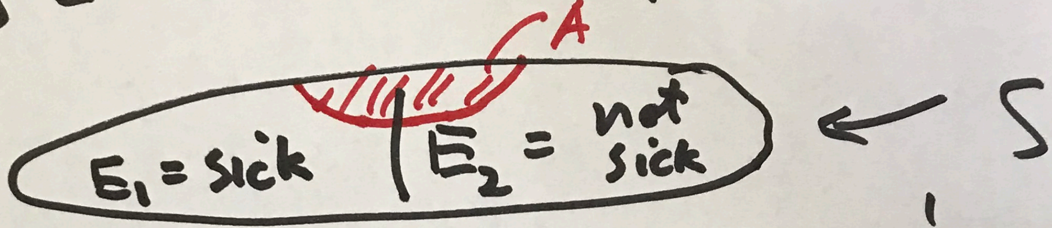
$E_1 \cap A, \ldots, E_n \cap A$ = disjoint partition of A

$$\underline{P}(A) = \underline{P}(E_1 \cap A) + \cdots + \underline{P}(E_n \cap A)$$

$$= \underline{P}(A|E_1)\,\underline{P}(E_1) + \cdots + \underline{P}(A|E_n)\,\underline{P}(E_n)$$

Ex: Medical Tests

S = all members of a population



A = the event that a person's medical test comes back <u>positive</u>

What is $P(E_1 \mid A)$??

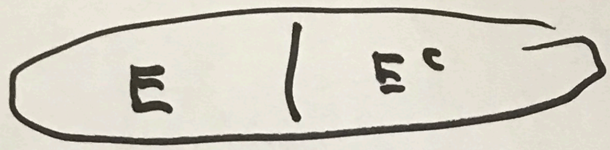
$$= \frac{P(A \mid E_1)\, P(E_1)}{P(A \mid E_1) P(E_1) + P(A \mid E_2)\, P(E_2)}$$

$$= \frac{(0.9)(0.001)}{(0.9)(0.001) + (0.01)(0.999)} \cong \frac{1}{12}$$

False Positives predominate.

Example

S = Pop. of New York City



E = people with fever today.

A = people showing coronavirus symptoms.

$$P(A|E) = ?? = \frac{P(E|A)\,P(A)}{P(E|A)P(A) + P(E|A^c)P(A^c)}$$

$$P(A) = \frac{1}{700}, \quad P(A^c) = 1 - \frac{1}{700}$$

$$P(E|A) = 1, \quad P(E|A^c) \simeq 1.5\cdot 10^{-3}$$

$P(A|E) = .488$. If $P(E|A^c) = 1.5\cdot 10^{-4}$

$$\frac{1/700}{1/700 + (1.5\cdot 10^{-4})(1 - \frac{1}{700})} = \frac{1}{1 + 1.5\cdot 10^{-4}\cdot 699} \simeq 0.9$$