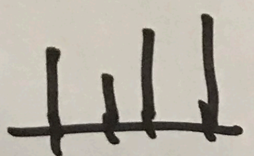
Expectations of Random Variables

$$\underline{X} : S \to \mathbb{R}$$

Discrete: S = finite

$$f_{\underline{X}}(r) = \text{Prob}(\underline{X} = r)$$



$$E(\underline{X}) = \text{expectation of } X$$

$$= \sum_{\substack{\text{all values} \\ r_i \text{ of } \underline{X}}} r_i \, f_{\underline{X}}(r_i)$$

Continuous Case:

$$f_{\underline{X}} : \mathbb{R} \to \mathbb{R} \quad \text{continuous}$$

$$\text{Prob}(a \le \underline{X} \le b) = \int_{t=a}^{t=b} f(t)\,dt$$

$$E(\underline{X}) = \int_{t=-\infty}^{t=\infty} t\, f(t)\,dt$$

__Example__: Fix $n \geq 1$

$S = \{ A = (A_1, \ldots, A_n) ; \; A_i = 0 \text{ or } 1 \}$

For each $i \in \{1, \ldots, n\}$ have

$\underline{X}_i : S \to \{0,1\} \subseteq \mathbb{R}$

$\underline{X}_i((A_1, \ldots, A_n)) = A_i$.

$f_{\underline{X}_i} : \mathbb{R} \to [0,1]$

$$f_{\underline{X}_i}(r) = \begin{cases} 0 & \text{if } r \notin \{0,1\} \\ p_i & \text{if } r = 1 \\ 1-p_i & r = 0 \end{cases}$$

$$E(\underline{X}_i) = \sum_{r=0,1} r \cdot f_{\underline{X}_i}(r)$$

$$= 0 \cdot (1-p_i) + 1 \cdot p_i = p_i$$

Here $X_i : S \longrightarrow \{0,1\}$

$$S = \{ s = (s_1, \dots, s_n) \; ; \; s_i \in \{0,1\} \}$$

$$X = X_1 + \dots + X_n : S \to \{0, \dots, n\} \subseteq \mathbb{R}$$

$$X((s_1, \dots, s_n)) = \#\{ i \; ; \; s_i = 1 \}$$

$$f_X(k) = \text{Prob}(X = k)$$

= Prob of getting exactly k "heads" = "ones".

<u>Theorem</u>: Given Random Variables

$$X_1, \dots, X_n : S \to \mathbb{R}$$

one has

$$E(X_1 + \dots + X_n) = E(X_1) + \dots + E(X_n)$$

<u>Example</u>: $E(X_i) = p_i$

$E(X_1 + \cdots + X_n) = p_1 + \cdots + p_n$ in coin flip example.

$S = \{(a_1, \ldots, a_n) \; ; \; a_i = 0 \text{ or } 1\}$

<u>Sketch of Proof</u> (Discrete case)

Joint density function of

$X_1, \ldots, X_n : S \to \mathbb{R}$.

Discrete case $f : \mathbb{R}^n \to [0, 1]$

so for all $r_1, \ldots, r_n \in \mathbb{R}$,

$\text{Prob}(X_1 = r_1, X_2 = r_2, \ldots, X_n = r_n)$

$= f(r_1, \ldots, r_n)$

Continuous Case:

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$\text{Prob}(a_1 \le \underline{X}_1 \le b_1, \dots, a_n \le \underline{X}_n \le b_n)$$

$$= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(t_1, \dots, t_n)\, dt_1 \dots dt_n$$

Important Fact:

Discrete Case

$$f_{\underline{X}_1}(r_1) = \sum_{r_2, \dots, r_n \in \mathbb{R}} f(r_1, r_2, \dots, r_n)$$

$$= \sum_{r_2, \dots, r_n \in \mathbb{R}} \text{Prob}\left(\begin{array}{l} \underline{X}_1 = r_1 \text{ and} \\ \underline{X}_2 = r_2 \dots \underline{X}_n = r_n \end{array}\right)$$

$$= \text{Prob}(\underline{X}_1 = r_1)$$

Proof that if $\underline{X} = \underline{X}_1 + \cdots + \underline{X}_n$ then

$E(\underline{X}) = E(\underline{X}_1) + \cdots + E(\underline{X}_n)$

$f = f_{\underline{X}}$ joint density for $\underline{X}$.

$$E(\underline{X}) = \sum_{r \in \mathbb{R}} \mathrm{Prob}(\underline{X} = r) \cdot r$$

$$= \sum_{r_1, \ldots, r_n \in \mathbb{R}} \mathrm{Prob}(\underline{X}_1 = r_1, \ldots, \underline{X}_n = r_n) \cdot (r_1 + \cdots + r_n)$$

$$= \sum_{i=1}^{n} \sum_{r_i \in \mathbb{R}} r_i \cdot \sum_{\substack{\text{all choices} \\ \text{of } r_j \text{ for } j \neq i}} \mathrm{Prob}(\underline{X}_1 = r_1, \ldots, \underline{X}_n = r_n)$$

$$= \sum_{i=1}^{n} \left(\sum_{r_i \in \mathbb{R}} r_i f_{\underline{X}_i}(r_i) \right)$$

$$= \sum_{i=1}^{n} E(\underline{X}_i)$$

Ex: n Independent flips of a coin which has prob. p of coming up heads

$$S = \{(t_1, \dots, t_n);\ t_i = 0 \text{ or } 1\}.$$

$$X_i : \downarrow \{0,1\} \qquad E(X_i) = p \text{ all } i$$

$$E(\underbrace{X_1 + \dots + X_n}_{X}) = \sum_{i=1}^{n} E(X_i) = np$$

$$f_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E(X) = \sum_{k=0}^{n} f_X(k) \cdot k = np$$

$$\parallel$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot k$$

Connection To Epidemics

S = all people who now would test positive for covid-19.

Historical

$$\underline{X} : S \longrightarrow \mathbb{Z}_{\geq 0} \subseteq \mathbb{R}$$

$\underline{X}(A)$ = how many other people are infected by person A.

Projected

$$\underline{X} : S \rightarrow \mathbb{R}_{\geq 0}$$

$\underline{X}(A)$ = estimated for the expected number of new infections resulting from A.

Important:

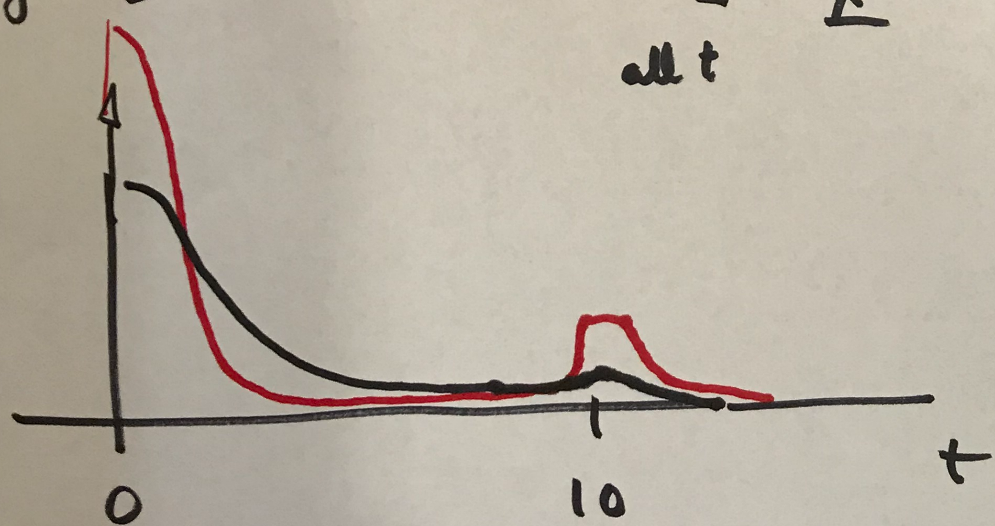
$E(X) =$ average (expected) number of new infections caused by each person who now has covid-19.

Projected Case: $X : S \to \mathbb{R}_{\geq 0}$

f_X = density

$$\int_{\text{all } t} f_X(t)\,dt = 1$$

$y = f_X(t)$



$$E(X) = \int_{t=0}^{\infty} t\, f_X(t)\,dt$$

$\downarrow$

$E(X)$