RANDOM VARIABLES

S = sample space

$\mathcal{B}$ = σ-algebra of subsets E of S.

$\underline{P} : \mathcal{B} \to [0,1]$ prob. function

$$\begin{array}{ccc} \mathcal{B} & \to & [0,1] \\ \cup\!\!\!\shortmid & & \\ E & \to & \underline{P}(E) \end{array}$$

Def: A random variable is a function $\mathbb{X} : S \to \mathbb{R}$ so for all $r \in \mathbb{R}$,

$$\underline{E}(r) = \{ s \in S \,;\, \mathbb{X}(s) \leq r \}$$

is in $\mathcal{B}$.

$\text{Prob}(\mathbb{X} \leq r) = \underline{P}(E(r))$ is well defined for all r in $\mathbb{R}$.

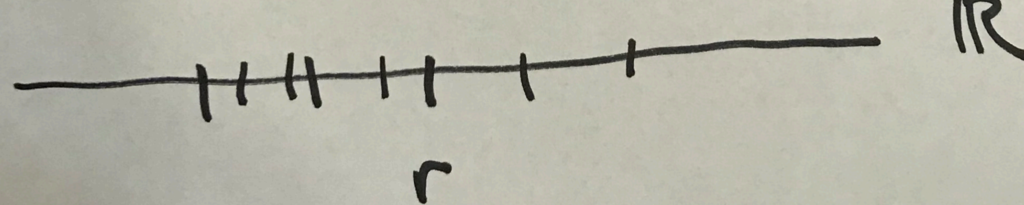
__Example__ $S = \{A_1, \dots, A_m\}$ = finite

$\mathcal{B} = 2^S$ = all subsets

$\underline{X} : S \to \mathbb{R}$

$r \in \mathbb{R}$ $E(r)$ = all A_i with $\underline{X}(A_i) \le r$

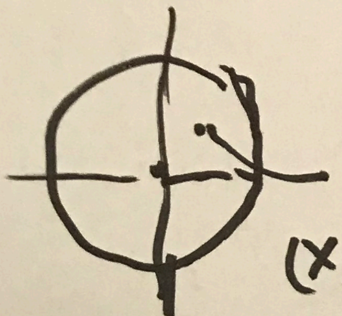
__Note__: $\underline{X}$ takes finitely many values



If $\underline{P} : \mathcal{B} \to [0,1]$, $\underline{P}(\{A_i\}) = \frac{1}{\# S}$

$$\underline{P}\text{rob}(\underline{X} \le r) = \frac{\#\{A \in S ; \underline{X}(A) \le r\}}{\# S}$$

Example: S = unit disc in $\mathbb{R}^2$



(x, y)

$\mathcal{B}$ = σ-alg. of Borel subsets of S

= smallest σ-alg. of subsets that contains all closed sets

$$\mathbb{X} : S \to \mathbb{R}$$
$$(x, y) \to \sqrt{x^2 + y^2}$$

$$\text{Prob}(\mathbb{X} \leq r) = \text{Prob}(E(r))$$

$$E(r) = \left\{ (x, y) \in S \; ; \; \sqrt{x^2+y^2} \leq r \right\}.$$

If: $\text{Prob}(E) = \text{Area}(E) / \text{Area}(S)$

Then $\text{Prob}(\mathbb{X} \leq r) = \dfrac{\pi r^2}{\pi} = r^2$

for $0 \leq r \leq 1$

Probability Density Functions + Distribution Functions

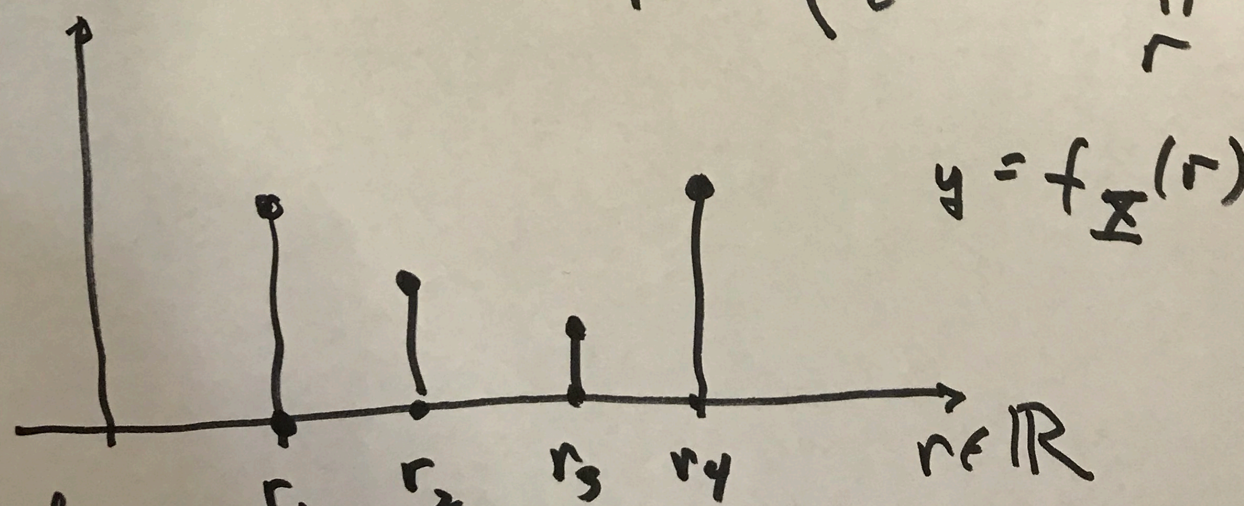
Discrete Case: $S = \text{finite}$

$$\mathbb{X} : S \to \mathbb{R}$$

Prob. Density function of $\mathbb{X} = f_{\mathbb{X}}$

For all r in $\mathbb{R}$

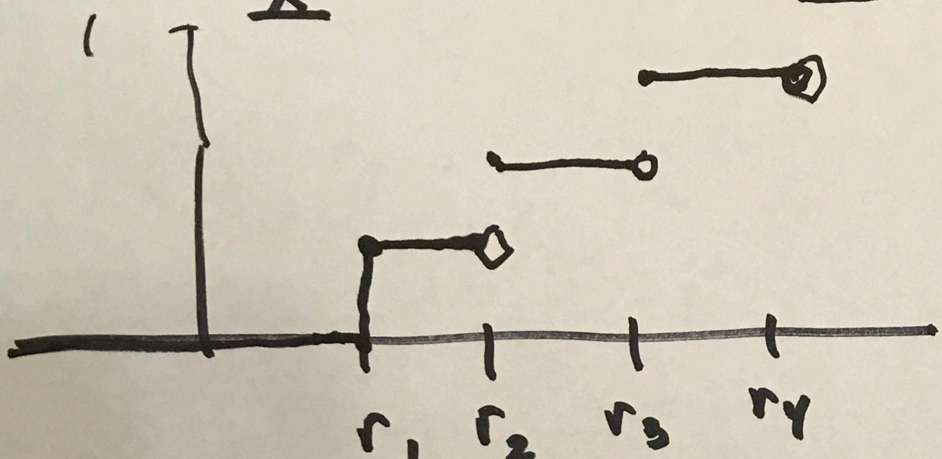
$$f_{\mathbb{X}}(r) = \text{Prob}(\mathbb{X} = r) = \text{Prob}(\{s \in S : \underbrace{X(s)}_{= r}\})$$



$$\sum_{k=1}^{l} f_{\mathbb{X}}(r_k) = P(S) = 1$$

Distribution Function of $X : S \to \mathbb{R}$

$$F_X(r) = \underline{P}rob(X \le r)$$



$r \in \mathbb{R}$

EX: Binomial Random Variable.

S = set of all ordered sequences of n independent trials,
prob. of each succeeding is p
prob. of not succeeding is $1-p$

$X : S \to \mathbb{R}$

$f_X(k)$ $\quad (A_1, \dots, A_n) \to$ # of successes ("numbers of heads")

$$\underline{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } 0 \le k \le n$$

Continuous Probability Density Function's

S = arbitrary sample space

$\underline{X} : S \to \mathbb{R}$

__Def__: $\underline{X}$ has continuous prob. density function $f_{\underline{X}} : \mathbb{R} \to \mathbb{R}$ if for all $a \le b$,

$$\mathrm{Prob}(a \le \underline{X} \le b) = \int_a^b f_{\underline{X}}(t)\,dt$$

$$\|$$

$$P(\{s \in S ; a \le \underline{X}(s) \le b\})$$

Distribution Function:

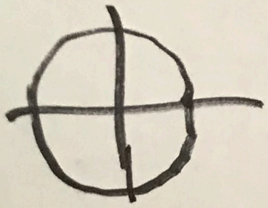
Continuous $F_{\underline{X}} : \mathbb{R} \to \mathbb{R}$ so

$$\mathrm{Prob}(\underline{X} \le r) = F_{\underline{X}}(r) = \int_{-\infty}^{r} f_{\underline{X}}(t)\,dt$$

Example: $S = \text{unit disc} \subseteq \mathbb{R}^2$

$\rotatebox{90}{$\subseteq$}$

E



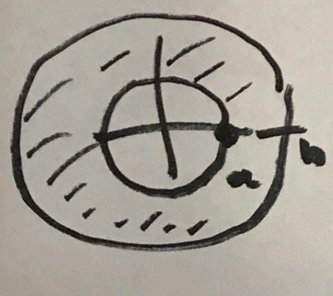
$$\mathbb{P}(E) = \frac{\text{Area}(E)}{\text{Area}(S)}$$

$$X : S \to \mathbb{R}$$

$$(x,y) \to \sqrt{x^2+y^2}$$

$$\text{Prob}(a \le X \le b) = \frac{\text{Area of annulus}}{\text{Area}(S)}$$

$0 \le a \le b \le 1$



$$= \frac{\int_{t=a}^{t=b} 2\pi t \, dt}{\pi} = \frac{\pi b^2 - \pi a^2}{\pi}$$

$$= \int_{t=a}^{t=b} 2t \, dt$$

$$f_X(t) = \begin{cases} 2t & \text{if } 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$