NOTES ON AUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS

APRIL 2013

These notes give a quick summary of the part of the theory of autonomous ordinary differential equations relevant to modeling zombie epidemics.

1. AUTONOMOUS LINEAR DIFFERENTIAL EQUATIONS, EQUILIBRIA AND STABILITY

Suppose that \( n \geq 1 \). We are going to later multiply vectors of length \( n \) on the left by square matrices. So we will consider vectors of length \( n \) to be column vectors

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.
\]

This will lessen the need to take transposes of row vectors as in the lectures in class.

Let \( G : \mathbb{R}^n \to \mathbb{R}^n \) is a continuously differentiable vector valued function. Thus \( G \) assigns to each \( x \in \mathbb{R}^n \) a vector

\[
G(x) = \begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_n(x) \end{pmatrix}
\]

and all the partial derivatives \( \frac{\partial G_i}{\partial x_j} \) are continuous functions of \( x \).

**Definition 1.1.** An autonomous ordinary differential equation for a function \( f : \mathbb{R} \to \mathbb{R}^n \) has the form

\[
\frac{df}{dt}(t) = G(f(t)) \quad \text{and} \quad f(0) = x
\]

in which \( f : \mathbb{R} \to \mathbb{R}^n \) is a function and \( x \) is an initial value for \( f \). Here

\[
f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}
\]

for some functions \( f_1, \ldots, f_n \) from \( \mathbb{R} \) to \( \mathbb{R} \) and

\[
\frac{df}{dt}(t) = \begin{pmatrix} \frac{df_1}{dt}(t) \\ \frac{df_2}{dt}(t) \\ \vdots \\ \frac{df_n}{dt}(t) \end{pmatrix}.
\]
The differential equation (1.1) is called autonomous because \( \frac{df}{dt} \) depends only on what \( f(t) \) is rather than depending on both \( f(t) \) and the independent variable \( t \). If one thinks of \( t \) as time and \( f(t) \) as the state of a system at time \( t \), then the rate of change \( \frac{df}{dt} \) of the system depends only on its state \( f(t) \) at time \( t \), not on what time it is.

**Definition 1.2.** The initial value \( x \) gives an equilibrium solution if \( f(t) = x \) for all \( t \) is a solution of the differential equation (1.1).

The following result is easy:

**Lemma 1.3.** The vector \( x \) gives an equilibrium solution \( f(t) = x \) for all \( t \) if and only if \( G(x) \) is the zero vector \( \mathbf{0} = (0, \ldots, 0) \) transpose.

**Proof.** If \( f(t) = x \) for all \( t \) is a solution to the differential equation, we have

\[
\frac{df}{dt}(t) = 0 = G(f(t)) = G(x)
\]

for all \( t \).

Conversely, if \( G(x) = \mathbf{0} \), and we let \( f(t) = x \) for all \( t \), then (1.2) holds, so \( f(t) \) solves the differential equation. \( \square \)

**Definition 1.4.** An equilibrium solution \( f(t) = x \) for all \( t \) is stable if there an \( \epsilon > 0 \) such that the following is true for all \( \tilde{x} \in \mathbb{R}^n \) such that distance(\( x, \tilde{x} \)) < \( \epsilon \). Suppose \( \tilde{f}(x) \) is a solution of the differential equation

\[
\frac{d\tilde{f}}{dt}(t) = G(\tilde{f}(t))
\]

with initial condition \( \tilde{f}(0) = \tilde{x} \). Then

\[
\lim_{t \to +\infty} \tilde{f}(t) = x.
\]

2. Linearizing the Differential Equation, and Linear Stability

To study stability near an equilibrium \( x \), we use the Taylor expansion of \( G(\tilde{x}) \) for \( \tilde{x} \) near \( x \). The Taylor expansion is

\[
G(\tilde{x}) = G(x) + \text{Jac}(G)(x) \cdot (\tilde{x} - x) + \text{higher order terms}.
\]

Here \( \text{Jac}(G) \) is the \( n \times n \) matrix

\[
\left( \frac{\partial G_i}{\partial x_j} \right)_{1 \leq i,j \leq n}
\]

and \( \text{Jac}(G)(x) \) is the matrix of constants which results from evaluating at \( x \) all the partial derivatives which are the entries of \( \text{Jac}(G) \). The higher order terms in the expansion go to 0 more rapidly than any linear function of \( \tilde{x} - x \) and \( \tilde{x} \) tends toward \( x \).

If \( \tilde{f}(t) \) were a solution of the differential equation such that \( \tilde{f}(t) \) is always close to \( x \), then since \( G(x) = \mathbf{0} \) for an equilibrium value of \( x \), we would have

\[
G(\tilde{f}(t)) = \text{Jac}(G)(x) \cdot (\tilde{f}(t) - x) + \text{smaller order error}.
\]

We use the first term on the right side of this expression to get a linear approximation to our original differential equation:
Definition 2.1. The linear approximation to the differential equation \( \frac{df}{dt}(t) = G(\tilde{f}(t)) \) near an equilibrium value \( x \) is
\[
(2.3) \quad \frac{df}{dt}(t) = A \cdot (\tilde{f}(t) - x)
\]
When \( A \) is the constant \( n \times n \) matrix \( A = \text{Jac}(G)(x) \).

Lemma 2.2. Let \( x \) be an equilibrium solution of (2.3). Then \( x \) is a stable equilibrium for (2.3) if and only if every solution \( y(t) \) of the differential equation
\[
(2.4) \quad \frac{dy}{dt}(t) = Ay(t)
\]
has the property that \( \lim_{t \to +\infty} y(t) = 0 = (0, \ldots, 0)^{\text{transpose}} \), where \( A = \text{Jac}(G)(x) \).

Proof. Suppose first that all \( y(t) \) as in the Lemma have \( \lim_{t \to +\infty} y(t) = 0 \). By definition, \( x \) is a stable equilibrium solution of (2.3) if and only for all \( \tilde{f}(t) \) for which (2.3) holds and \( \tilde{f}(0) = \tilde{x} \) is sufficiently close to \( x \), we have \( \lim_{t \to +\infty} \tilde{f}(t) = x \). Set \( y(t) = \tilde{f}(t) - x \). Then \( \frac{dy}{dt} = \frac{df}{dt} \), so \( y(t) \) is a solution of (2.4). Thus
\[
\lim_{t \to +\infty} \tilde{f}(t) = \lim_{t \to +\infty} (y(t) + x) = 0 + x = x
\]
and (2.3) has \( x \) as a stable equilibrium solution.

Conversely, suppose that \( x \) is a stable equilibrium solution to (2.3), and that \( y(t) \) is a solution of (2.4). By definition, this means that there is an \( \epsilon > 0 \) such that if distance \((x, \tilde{x}) < \epsilon\), then every solution \( \tilde{f} \) of (2.3) such that \( \tilde{f}(0) = \tilde{x} \) has the property that
\[
\lim_{t \to +\infty} \tilde{f}(t) = x.
\]
We can find a real constant \( c > 0 \) such that distance \((0, c \cdot y(0)) < \epsilon\). Since \( y(t) \) is a solution of (2.4), we find that \( \tilde{f}(t) = x + c \cdot y(t) \) is a solution of (2.3) since
\[
\frac{d\tilde{f}}{dt}(t) = c \cdot \frac{dy}{dt}(t) = c \cdot Ay(t) = Ac \cdot y(t) = A(\tilde{f}(t) - x).
\]
We have
\[
\text{distance}(\tilde{f}(0), x) = \text{distance}(x + c \cdot y(0), x) = \text{distance}(cy(0), 0) < \epsilon.
\]
Therefore \( \lim_{t \to +\infty} \tilde{f}(t) = x \) since \( x \) is a stable equilibrium, so
\[
\lim_{t \to +\infty} y(t) = \lim_{t \to +\infty} (\tilde{f}(t) - x)/c = 0.
\]
\( \square \)

Definition 2.3. Let \( x \) be an equilibrium solution of the original differential equation (1.1). We will say that \( x \) is a linearly stable equilibrium if the linearized differential equation (2.1) has \( x \) as a stable solution. By Lemma 2.2, this is equivalent to requiring that \( \lim_{t \to +\infty} y(t) = 0 \) for all solutions \( y(t) \) to
\[
\frac{dy}{dt}(t) = Ay(t)
\]
when \( A = \text{Jac}(G)(x) \).
3. Solutions of linear systems

Suppose $A$ is an $n \times n$ matrix of complex numbers.

**Theorem 3.1.** The series of $n \times n$ matrices

$$e^A = \sum_{m=0}^{\infty} \frac{(At)^m}{m!}$$

converges for all values of $t$ to an $n \times n$ matrix of complex numbers. The unique solution $y : \mathbb{R} \to \mathbb{R}^n$ of the differential equation

$$(3.5) \quad \frac{dy}{dt}(t) = Ay(t) \quad \text{and} \quad y(0) = w$$

is

$$y(t) = e^{At}w.$$ 

This result does take some time to prove rigorously, so I will not include a proof. The appearance of $e^{At}$ is justified by the following formal computation:

$$\frac{d}{dt} e^{At}w = \frac{d}{dt} \sum_{m=0}^{\infty} \frac{A^m t^m}{m!}w$$

$$= \sum_{m=0}^{\infty} \frac{d}{dt} \frac{A^m t^m}{m!}w$$

$$= \sum_{m=1}^{\infty} \frac{A^m m t^{m-1}}{m!}w$$

$$= \sum_{m=1}^{\infty} A^{m-1} \frac{t^{m-1}}{(m-1)!}w$$

$$= Ae^{At}w.$$

$$(3.6)$$

The following result is proved in linear algebra courses.

**Theorem 3.2.** There is an invertible $n \times n$ matrix $B$ such that $C = BAB^{-1}$ has the upper triangular form

$$C = \begin{pmatrix}
\lambda_1 & c_{1,2} & c_{1,3} & \cdots & c_{1,n} \\
0 & \lambda_2 & c_{2,3} & \cdots & c_{2,n} \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & \lambda_3 & \cdots & c_{3,n} \\
0 & 0 & 0 & 0 & \cdots & \lambda_n
\end{pmatrix}$$

$$(3.7)$$

for some complex numbers $\lambda_1, \cdots, \lambda_n$ and $c_{i,j}$ with $i < j$. The entries below the diagonal of $C$ are all 0. The numbers $\lambda_i$ for $1 \leq i \leq n$ may not all be distinct, and are called the eigenvalues of $A$. These $\lambda_i$ are the roots (counting multiplicities) of the characteristic polynomial

$$c_A(T) = \det(T \cdot I_n - A)$$

where $T$ is an indeterminate and $I_n$ is the $n \times n$ identity matrix.

Observe now that if $m \geq 0$, then

$$C^m = (BAB^{-1}) = (BAB^{-1}) \cdot (BAB^{-1}) \cdots (BAB^{-1}) = BA^m B^{-1}$$
since we can group terms in the product to take advantage of the fact that \(BB^{-1} = I_n\). Thus we get

\[
Be^{At}B^{-1} = B \left( \sum_{m=0}^{\infty} \frac{(At)^m}{m!} \right) B^{-1}
\]

\[
= \sum_{m=0}^{\infty} \frac{BA^m B^{-1} t^m}{m!}
\]

\[
= \sum_{m=0}^{\infty} \frac{C^m t^m}{m!}
\]

(3.8)

Since \(C\) in (3.7) is upper triangular, the matrix \(C^m\) is also upper triangular, with diagonal terms given by \(\lambda_1^m, \ldots, \lambda_n^m\). Thus

\[
e^{Ct} = \sum_{m=0}^{\infty} \frac{C^m t^m}{m!}
\]

\[
= \begin{pmatrix}
\sum_{m=0}^{\infty} \frac{\lambda_1^m t^m}{m!} & \ast & \ast & \cdots & \ast \\
0 & \sum_{m=0}^{\infty} \frac{\lambda_2^m t^m}{m!} & \ast & \cdots & \ast \\
0 & 0 & \sum_{m=0}^{\infty} \frac{\lambda_3^m t^m}{m!} & \cdots & \ast \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \sum_{m=0}^{\infty} \frac{\lambda_n^m t^m}{m!}
\end{pmatrix}
\]

(3.9)

where we don’t need to calculate the entries above the diagonal.

Suppose now that \(x\) is a linearly stable equilibrium solution of the original differential equation (1.1) in the sense of Definition 2.3. Let \(A = \text{Jac}(G)(x)\). Then for all initial values \(y(0) = w\) of the differential equation

\[
\frac{dy}{dt}(t) = Ay(t)
\]

we should have \(\lim_{t \to +\infty} y(t) = 0\). By Theorem 3.1, this is equivalent to

\[
\lim_{t \to +\infty} e^{At}w = 0.
\]

Taking \(w\) to range over the vectors which have exactly one component equal to 1 and all the others equal to 0, we would be able to conclude that

\[
\lim_{t \to +\infty} e^{At} = \text{the zero matrix}.
\]

Then (3.8) would give

\[
\lim_{t \to +\infty} e^{Ct} = B \left( \lim_{t \to +\infty} e^{At} \right) B^{-1} = \text{the zero matrix}.
\]
But now (3.9) would show

\[(3.10) \quad \lim_{t \to +\infty} e^{\lambda_i t} = 0\]

for \(i = 1, \ldots, n\). Here if \(\lambda_i = a_i + b_i \sqrt{-1}\) we have

\[e^{\lambda_i t} = e^{a_i t}(\cos(b_i t) + \sqrt{-1}\sin(b_i t)).\]

So (3.10) is equivalent to

\[(3.11) \quad \text{Re}(\lambda_i) = a_i < 0 \quad \text{for} \quad i = 1, \ldots, n.\]

With a little more work, which I won’t include, one can show that (3.11) is also sufficient for linear stability:

**Theorem 3.3.** The differential equation (1.1) is linearly stable at an equilibrium \(x\) if and only if the eigenvalues \(\lambda_1, \ldots, \lambda_n\) of the Jacobian matrix \(A = \text{Jac}(G)(x)\) at \(x\) have all negative real parts.