LINEAR PROGRAMMING PROBLEMS AND VERTICES

1. Statement of the Result

Suppose \( n, m \geq 1 \) are integers, \( B = (b_{ij})_{1 \leq i \leq n, 1 \leq j \leq m} \) is an \( n \times m \) matrix of positive numbers, and \( b = (b_1, \ldots, b_m) \) and \( c = (c_1, \ldots, c_n) \) are vectors of positive numbers. The linear programming problem we will consider is to find the set \( \mathcal{O} \) of all vectors \( s = (s_1, \ldots, s_n) \) which satisfy the constraints

\[
s \geq (0, \ldots, 0) \quad \text{and} \quad sB \geq b
\]

and which minimize the objective function

\[
f(s) = s \cdot c^{\text{transpose}} = s_1c_1 + \cdots + s_nc_n
\]

We call \( \mathcal{O} \) the set of solutions of the linear programming problem. We proved earlier (using the theorem that a continuous function on a closed bounded set) that all linear programming problems of this kind have a solution.

Define constants \( d_q \) and linear functions \( h_q(s) \) of \( s = (s_1, \ldots, s_n) \) for \( q = 1, \ldots, n + m \) in the following way. If \( 1 \leq q \leq n \) let \( h_q(s) = s_q \), and let \( d_q = 0 \). If \( q = n + j \) and \( 1 \leq j \leq m \), let \( h_q(s) = s_1b_{1,j} + s_2b_{2,j} + \cdots + s_nb_{n,j} \) and \( d_q = b_j \). Thus if \( q = n + j \), \( h_q(s) \) is just the \( j \)-th entry of the row vector \( sB \). We can therefore write the constraints (1.1) as

\[
h_q(s) \geq d_q \quad \text{for} \quad 1 \leq q \leq n + m
\]

Definition 1.1. A point \( s = (s_1, \ldots, s_n) \in \mathbb{R}^n \) is a vertex of the linear programming problem if there is a subset \( T \) of \( \{1, \ldots, n + m\} \) for which the following is true:

i. \( s \) is the only solution of the equalities \( h_q(s) = d_q \) for \( q \in T \).
ii. One has \( h_q(s) \geq d_q \) for all \( q = 1, \ldots, n + m \).

Definition 1.2. The convex hull \( C(P) \) of a set \( P \) of vectors in \( \mathbb{R}^n \) is the set of all linear combinations

\[
d_1p_1 + \cdots + d_\ell p_\ell
\]

of elements \( p_1, \ldots, p_\ell \) of \( P \) such that \( \ell \geq 1, 0 \leq d_i \in \mathbb{R} \) for all \( 1 \leq i \leq \ell \) and \( d_1 + \cdots + d_\ell = 1 \). (Note that we allow \( P \) to be infinite.)

Theorem 1.3. Let \( \mathcal{V} \) be the set of vertices of the linear programming problem. Define \( \mathcal{P} = \{p_1, \ldots, p_\ell\} \) be the set of \( s \in \mathcal{V} \) for which \( f(s) = \min\{f(s'): s' \in \mathcal{V}\} \). Then the set \( \mathcal{O} \) of all solutions of the linear programming problem is the convex hull \( C(\mathcal{P}) \) of \( \mathcal{P} \).

2. Solutions which are not vertices

The proof of Theorem 1.3 depends on the following Proposition.

Proposition 2.1. For \( s \in \mathbb{R}^n \), define \( T(s) \) to be the set of \( q \) in the range \( 1 \leq q \leq n + m \) such that \( h_q(s) = d_q \). Suppose \( s \in \mathcal{O} \) is a solution of the linear programming problem which is not a vertex. Then there are points \( s' \) and \( s'' \) in \( \mathcal{O} \) with the following properties:

i. \( s \) lies on the line segment between \( s' \) and \( s'' \), so that \( s = \alpha s' + (1 - \alpha) s'' \) for some \( 0 \leq \alpha \leq 1 \), and
ii. The sets \( T(s') \) and \( T(s'') \) contain \( T(s) \) and are both strictly larger than \( T(s) \).
To prove this we need some lemmas. Observe first that since $s$ is not a vertex, and it satisfies all the constraints of the linear programming problem, it cannot be the unique element of $\mathbb{R}^n$ which is a solution of $h_q(s) = d_q$ for $q \in T(s)$. So there is another point $\tilde{s} \in \mathbb{R}^n$ such that $\tilde{s} \neq s$ and $h_q(\tilde{s}) = d_q = h_q(s)$ for $q \in T(s)$.

**Definition 2.2.** For $1 \leq q \leq n + m$ let $W_q(s)$ be the set of all real numbers $t$ for which $s + t(\tilde{s} - s)$ satisfies $h_q(s + t(\tilde{s} - s)) \geq d_q$.

**Lemma 2.3.** Suppose $1 \leq q \leq n + m$.

i. If $h_q(\tilde{s}) = h_q(s)$ then $W_q = \mathbb{R}$.

ii. If $h_q(\tilde{s}) > h_q(s)$ then $W_q = [-a_q, \infty)$ for some $a_q > 0$.

iii. If $h_q(\tilde{s}) < h_q(s)$ then $W_q = (-\infty, b_q]$ for some $b_q > 0$.

**Proof.** Since $h_q(s)$ is linear in $s$, $W_q(s)$ is the set of $t \in \mathbb{R}$ such that

\[
(2.4) \quad h_q(s + t(\tilde{s} - s)) = h_q(s) + t(h_q(\tilde{s}) - h_q(s)) \geq d_q.
\]

Here $h_q(s) = h_q(\tilde{s}) = d_q$ if $q \in T(s)$ and $h_q(s) > d_q$ if $1 \leq q \leq n + m$ and $q \notin T(s)$. We conclude that if $h_q(\tilde{s}) = h_q(s)$ then (2.4) holds for all $t \in \mathbb{R}$, which shows (i). In particular, this is so if $q \in T(s)$.

Suppose now that $q \notin T(s)$ and that $h_q(\tilde{s}) \neq h_q(s)$. If $h_q(\tilde{s}) > h_q(s)$, then the inequality (2.4) is

\[
t \geq (d_q - h_q(s))/(h_q(\tilde{s}) - h_q(s)).
\]

Here $h_q(s) > d_q$ since $q \notin T(s)$, so

\[
a_q = (h_q(s) - d_q)/(h_q(\tilde{s}) - h_q(s)) > 0
\]

and the inequality (2.4) is equivalent to $t \geq -a_q$. This shows (ii).

To show (iii), suppose $h_q(\tilde{s}) < h_q(s)$. Then $h_q(\tilde{s}) - h_q(s) < 0$, and (2.4) is equivalent to

\[
t \leq (h_q(s) - d_q)/(h_q(s) - h_q(\tilde{s})) = b_q.
\]

Here $b_q > 0$ because $q \notin T(s)$ implies $h_q(s) > d_q$. This establishes (iii). \hfill $\Box$

**Corollary 2.4.** The set

\[ W(s) = \bigcap_{q=1}^{n+m} W_q(s) \]

is the set of all real numbers $t$ such that $s + t(\tilde{s} - s)$ satisfies all the constraints of the linear programming problem. The set $W(s)$ is finite intersection of copies of $\mathbb{R}$ or half infinite closed intervals, all of which contain 0 in their interior. So

\[
(2.5) \quad W = \mathbb{R} \quad \text{or} \quad W = (-\infty, b] \quad \text{or} \quad W = [-a, \infty) \quad \text{or} \quad W = [-a, b] \quad \text{for some} \quad a, b > 0
\]

**Lemma 2.5.** One has $f(s + t(\tilde{s} - s)) = f(s)$ for all $t \in \mathbb{R}$. Every $t \in W(s)$ gives and element $s + t(\tilde{s} - s)$ in the set $\mathcal{O}$ of solutions to the linear programming problem.

**Proof.** We assumed that $s$ is a solution to the linear programming problem. So $s$ minimizes $f(s) = c_1s_1 + \cdots + c_ns_n$ over all $s = (s_1, \ldots, s_n)$ which satisfy the constraints $h_q(s) \geq d_q$ for $1 \leq q \leq n + m$. Suppose $f(\tilde{s}) \neq f(s)$. Then there will be values of $t$ arbitrarly close to 0 such that

\[
f(s + t(\tilde{s} - s)) = f(s) + t(f(\tilde{s}) - f(s))
\]

is smaller than $f(s)$, since $f(\tilde{s}) - f(s) \neq 0$. But (2.5) shows that all $t$ sufficiently close to 0 give points $s + t(\tilde{s} - s)$ which satisfy all the constraints. So have a contradiction to the fact that $s$ must minimize $f(s)$ over all $s$ which satisfy the constraints. This shows $f(\tilde{s}) = f(s)$, so

\[
f(s + t(\tilde{s} - s)) = f(s) + t(f(\tilde{s}) - f(s)) = f(s) \quad \text{for all} \quad t \in \mathbb{R}.
\]

If $t \in W(s)$, then $s + t(\tilde{s} - s)$
satisfies all the constraints, so this proves \( s + t(\tilde{s} - s) \) also minimizes the function \( f \) and is a solution of the linear programming problem. \( \square \)

**Lemma 2.6.** Write \( s = (s_1, \ldots, s_n) \) and \( \tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n) \). Among the numbers \( \{\tilde{s}_i - s_i\}_{i=1}^n \) there must be at least one number which is positive and one which is negative.

**Proof.** Recall that we assumed that \( c = (c_1, \ldots, c_n) \) is a vector of positive numbers. Since \( \tilde{s} \neq s \), not all of the numbers \( \tilde{s}_i - s_i \) are 0 as \( i \) ranges over \( 1 \leq i \leq n \). If \( \tilde{s}_i - s_i \geq 0 \) for all \( i \), then \( \tilde{s}_i - s_i > 0 \) for at least one \( i \), and the fact that \( c_i > 0 \) for all \( i \) implies

\[
    f(\tilde{s}) - f(s) = c_1(\tilde{s}_1 - s_1) + \ldots + c_n(\tilde{s}_n - s_n) > 0.
\]

This contradicts Lemma 2.5. Similarly, if \( \tilde{s}_i - s_i \leq 0 \) for all \( i \), then \( \tilde{s}_i - s_i < 0 \) for at least one \( i \), and

\[
    f(\tilde{s}) - f(s) = c_1(\tilde{s}_1 - s_1) + \ldots + c_n(\tilde{s}_n - s_n) < 0
\]

which contradicts Lemma 2.5. The only other alternative is that Lemma 2.6 holds. \( \square \)

**Corollary 2.7.** The set \( W(s) \) in Corollary 2.4 has the form \( W(s) = [-a, b] \) for some real numbers \( a, b > 0 \).

**Proof.** We know from Lemma 2.6 that there is an integer \( i \) in the range \( 1 \leq i \leq n \) with \( \tilde{s}_i - s_i > 0 \). The \( i^{th} \) constraint function \( h_i(s) \) is just \( h_i(s) = s_i \) and the constraint is \( h_i(s) = s_i \geq 0 = d_i \). So there is a lower bound on the real numbers \( t \) such that

\[
    h_i(s) + t(\tilde{s} - s) = s_i + t(\tilde{s}_i - s_i) \geq 0 = d_i.
\]

Similarly, Lemma 2.6 shows there is an integer \( j \) in the range \( 1 \leq j \leq n \) such that \( \tilde{s}_j - s_j < 0 \), and this gives an upper bound on the real numbers \( t \) such that

\[
    h_j(s) + t(\tilde{s} - s) = s_j + t(\tilde{s}_j - s_j) \geq 0 = d_j.
\]

So there is both an upper and a lower bound on elements of the set \( W(s) \) for all \( t \) for which \( s + t(\tilde{s} - s) \) satisfies all the constraints. This shows that in fact, the alternative in (2.5) which holds is that \( W(s) = [-a, b] \) for some real numbers \( a, b > 0 \). \( \square \)

**Completion of the proof of Proposition 2.1.** Let \( W(s) = [-a, b] \) as in Corollary 2.7. Consider the vector

\[
    s' = s + (-a) \cdot (\tilde{s} - s).
\]

This satisfies all the constraints, since \(-a \in W\). Furthermore \( f(s') = f(s) \) by Lemma 2.5, and this minimizes the value of \( f \) over all points satisfying the constraints since \( s \in O \) is a solution of the linear programming problem. Thus \( s' \) is also an element of the set \( O \) of all solutions.

One has \( h_q(s') = h_q(s) \) if \( q \in T(s) \) since for such \( q \), \( h_q(\tilde{s}) = h_q(s) = d_q \) by our choice of \( \tilde{s} \). So \( T(s) \subseteq T(s') \) since \( T(s') \) is the set of \( q \) for which \( h_q(s') = d_q \).

If \( T(s) = T(s') \) then for each \( q = 1, \ldots, n + m \) which is not in \( T(s) \), \( h_q(s') > 0 \). There will then be an \( \epsilon > 0 \) so that for all such \( q \not\in T(s) \), \( h_q(s' - \epsilon(\tilde{s} - s)) > 0 \). Since \( h_q(\tilde{s} - s) = 0 \) for \( q \in T(s) \), we get \( h_q(s' - \epsilon(\tilde{s} - s)) = 0 \) if \( q \not\in T(s) \). Thus \( s' - \epsilon(\tilde{s} - s) \) satisfies all the constraints. But

\[
    s' - \epsilon(\tilde{s} - s) = s - (a + \epsilon)(\tilde{s} - s)
\]

so \(-a - \epsilon \in W(s) = [-a, b] \), which is impossible because \( \epsilon > 0 \). We conclude that \( T(s) = T(s') \) is impossible, so \( T(s') \) is strictly larger than \( T(s) \).

Similarly, \( s'' = a + b(\tilde{s} - s) \) is an element of \( O \) which has the property that \( T(s'') \) strictly contains \( T(s) \).
Now Lemma 2.5 shows \( f(s) = f(s') = f(s'') \), so since \( s \) was an optimal solution of the linear programming problem, so are \( s' \) and \( s'' \). In other words, \( s, s' \) and \( s'' \) are in \( \mathcal{O} \).

Finally, to show that \( s \) lies on the line segment between \( s' = s - a(\tilde{s} - s) \) and \( s'' = s + b(\tilde{s} - s) \), let us check that

\[
s = zs' + (1 - z)s'' \quad \text{when} \quad z = b/(a + b).
\]

We have

\[
zs' + (1 - z)s'' = \left( \frac{b}{a + b} \right) (s - a(\tilde{s} - s)) + \left( \frac{a}{a + b} \right) (s + b(\tilde{s} - s))
\]

\[
= \left( \frac{b + a}{a + b} \right) s + \left( \frac{-ba + ab}{a + b} \right) (\tilde{s} - s)
\]

\[
= s.
\]

(2.6)

3. Completion of the proof of Theorem 1.3.

We need a few more Lemmas.

**Lemma 3.1.** Suppose \( s \in \mathcal{O} \), so that \( s \) is a solution of the linear programming problem. Recall that \( T(s) \) is the set of \( q \) in the range \( 1 \leq q \leq n + m \) such that \( h_q(s) = d_q \). Let \( \#T(s) \) be the number of elements of \( T(s) \). Define \( m(s) \) to be the maximum of \( \#T(\hat{s}) - \#T(s) \) over all \( \hat{s} \in \mathcal{O} \) such that \( T(s) \subset T(\hat{s}) \). Then \( 0 \leq m(s) \leq n + m \), and \( s \) is a vertex if and only if \( m(s) = 0 \).

**Proof.** The fact that \( 0 \leq m(s) \leq n + m \) is clear from the fact that for all \( s \), \( T(s) \) is a subset of the integers \( q \) in the range \( 1 \leq q \leq n + m \). If \( m(s) = 0 \) then there do not exist \( \hat{s} \in \mathcal{O} \) for which \( T(s) \) properly contains \( T(\hat{s}) \). If \( s \) were not a vertex, this contradicts part (ii) of Proposition 2.1. So \( s \) is a vertex if \( m(s) = 0 \).

Conversely, suppose that \( s \) is a vertex. Then \( s \) is the unique point in \( \mathbb{R}^n \) such that \( h_q(s) = d_q \) for all \( q \in T(s) \). If \( \hat{s} \in \mathcal{O} \) has the property that \( T(\hat{s}) \subset T(s) \), then \( h_q(\hat{s}) = d_q \) for all \( q \in T(\hat{s}) \), so in particular this is so for all \( q \in T(s) \). But for the \( s \) unique solution of \( h_q(s) = d_q \) for all \( q \in T(s) \), in fact \( \hat{s} = s \) and then \( T(s) = T(\hat{s}) \). This shows that \( \#T(\hat{s}) - \#T(s) = 0 \) for all \( \hat{s} \in \mathcal{O} \) such that \( T(s) \subset T(\hat{s}) \), so \( m(s) = 0 \) if \( s \) is a vertex. \( \square \)

**Lemma 3.2.** For \( r \geq 0 \), let \( \mathcal{S}(r) \) be the set of \( s \in \mathcal{O} \) for which \( m(s) \leq r \). Then \( \mathcal{S}(r + 1) \) contains \( \mathcal{S}(r) \) and is contained in the convex hull \( C(\mathcal{S}(r)) \) of \( \mathcal{S}(r) \).

**Proof.** It’s clear that \( \mathcal{S}(r) \subset \mathcal{S}(r + 1) \). Suppose \( s \in \mathcal{S}(r + 1) \), so that \( m(s) \leq r + 1 \). If \( m(s) \leq r \), then \( s \) is already an element of \( \mathcal{S}(r) \), so that \( s \) lies in \( C(\mathcal{S}(r)) \). So we now suppose that \( m(s) = r + 1 \). Then \( m(s) > 0 \), so by Lemma 3.1, \( s \) is not a vertex. Therefore, Proposition 2.1 shows \( s \) lies on the line segment between two points \( s', s'' \in \mathcal{O} \) such that \( T(s) \) is a proper subset of each of \( T(s') \) and \( T(s'') \). We now show that \( m(s') < m(s) = r + 1 \). If \( m(s') \geq r + 1 \), then there must exist a point \( \hat{s} \in \mathcal{O} \) so \( T(s') \supset T(\hat{s}) \) and \( \#T(\hat{s}) - \#T(s') \geq r + 1 \). But then \( T(s) \subset T(s') \subset T(\hat{s}) \) and \( \#T(\hat{s}) - \#T(s) > \#T(\hat{s}) - \#T(s') \geq r + 1 \) because \( T(s') \) is strictly larger than \( T(s) \). This would mean \( m(s) > r + 1 \), which is a contradiction. So \( m(s') < m(s) = r + 1 \), and similarly \( m(s'') < m(s) = r + 1 \). Thus \( s', s'' \in \mathcal{S}(r) \), so since \( s \) is on the line segment between \( s' \) and \( s'' \) we conclude that \( s \in C(\mathcal{S}(r)) \). \( \square \)

**Lemma 3.3.** Suppose \( P \subset P' \) are subsets of \( \mathbb{R}^n \) and that \( P' \) is a subset of the convex hull \( C(P) \) of \( P \). Then \( C(P) = C(P') \).
Thus (3.7) shows that
\[ p' = p_1' + \cdots + p_{\ell}'. \]

Now each \( p_i' \) lies in \( P' \), and \( P' \) is contained in the convex hull \( C(P) \) of \( P \). So each \( p_i' \) can be written as
\[ p_i' = \sum_{j=1}^{\ell(j)} d_{i,j} p_{i,j} \]
for some \( \ell(j) \geq 1 \), constants \( d_{i,j} \geq 0 \) such that \( \sum_{j=1}^{\ell(j)} d_{i,j} = 1 \) and some vectors \( p_{i,j} \in P \). We conclude that
\[
(3.7) \quad p' = \sum_i d_i p_i' = \sum_i \left( d_i \cdot \left( \sum_j d_{i,j} p_{i,j} \right) \right) = \sum_{i,j} d_i d_{i,j} p_{i,j}.
\]

Here \( d_i d_{i,j} \geq 0 \) since \( d_i \geq 0 \) and \( d_{i,j} \geq 0 \), and we have
\[ \sum_{i,j} d_i d_{i,j} = \sum_i d_i \sum_j d_{i,j} = \sum_i d_i = 1. \]

Thus (3.7) shows that \( p' \) is in the convex hull of \( P \), so \( C(P') \subset C(P) \) and we are done. \( \square \)

**Corollary 3.4.** With the notations of Lemma 3.2 we have \( \mathcal{O} = C(S(r)) = C(S(0)) \) for all \( r \geq 0 \).

**Proof.** We show \( C(S(r)) = C(S(0)) \) for all \( r \geq 0 \) by induction on \( r \). This statement is clearly true for \( r = 0 \). So it is enough to show that if it is true for some \( r \geq 0 \), then it is true when \( r \) is replaced by \( r+1 \). Lemma 3.2 shows \( S(r) \subset S(r+1) \subset C(S(r)) \). So Lemma 3.3 proves \( C(S(r+1)) = C(S(r)) \), and now the induction hypothesis \( C(S(0)) = C(S(r)) \) implies \( C(S(0)) = C(S(r+1)) \). Finally, since \( m(s) \leq n + m \) for \( s \in \mathcal{O} \), we see that every \( s \in \mathcal{O} \) lies in \( S(n+m) \). Hence \( \mathcal{O} \subset S(n+m) \subset C(S(n+m)) = C(S(0)) \). On the other hand, every element of \( S(0) \) is in \( \mathcal{O} \). So \( \mathcal{O} \subset C(S(0)) \subset C(\mathcal{O}) \). Thus to finish the proof, it will suffice to show that \( \mathcal{O} = C(\mathcal{O}) \). This is the same as saying that every finite linear combination
\[ y = e_1 y_1 + \cdots + e_\ell y_\ell \]
of elements \( y_i \) of \( \mathcal{O} \) with coefficients \( e_i \geq 0 \) satisfying \( \sum_i e_i = 1 \) is again in \( \mathcal{O} \). Since the \( y_i \) are in \( \mathcal{O} \) they satisfy the constraints \( h_q(y_i) \geq d_q \) for \( q = 1, \ldots, n+m \). Hence
\[ h_q(y) = \sum_i e_i h_q(y_i) \geq \sum_i e_i d_q = (\sum_i e_i) d_q = d_q \]
and therefore \( y \) satisfies the constraints. Since all the \( y_i \) minimize the value of the objective function \( f \), the value \( f(y_i) \) equals \( f(y_1) \) for all \( i \). Thus
\[ f(y) = \sum_i e_i f(y_i) = \sum_i e_i f(y_1) = f(y_1) \]
and this proves \( y \) is also a solution of the linear programming problem, i.e. that \( y \in \mathcal{O} \). \( \square \)

**Lemma 3.5.** Recall that \( V \) is the (finite) set of vertices of the linear programming problem. The set \( S(0) \) is exactly the set \( P \) of all \( s \in V \) such that \( s = \min \{ f(s') : s' \in V \} \).
Proof. By definition, \( S(0) \) is the set of solutions \( s \in O \) for which \( m(s) = 0 \), and Lemma 3.1 shows these are exactly the \( s \) in \( O \) which are also in \( V \). Since every element of \( V \) satisfies the constraints, and \( s \in O \) is a solution of the linear programming problem, we must have \( f(s) \leq f(s') \) for all \( s' \in V \). Thus \( S(0) = O \cap V \subset P \).

We now have to show \( P \subset S(0) \). Since every element of \( P \) is a vertex, this is the same as showing that every element of \( P \) is in \( O \), i.e., that every element of \( P \) solves the linear programming problem. Each element of \( P \) satisfies all the constraints. So it will be enough to show that if \( \hat{s} \in P \) and \( s \) is a solution of the linear programming problem, then \( f(s) \geq f(\hat{s}) \), since then \( \hat{s} \) must also minimize the value of \( f \) and \( f(\hat{s}) = f(s) \).

We proved in Corollary 3.4 that \( s \) lies in the convex hull \( C(S(0)) \). So

\[
s = e_1 y_1 + \cdots + e_\ell y_\ell
\]

for some \( y_i \in S(0) \subset V \) and some \( e_i \geq 0 \) such that \( \sum_i e_i = 1 \). Thus

\[
f(s) = \sum_i e_i f(y_i) \geq (\sum_i e_i) \cdot \min_i f(y_i) = \min_i f(y_i) \geq \min\{f(v) : v \in V\}.
\]

But \( \hat{s} \in P \) means that

\[
f(\hat{s}) = \min\{f(v) : v \in V\}
\]

so we get

\[
f(s) \geq f(\hat{s})
\]

which completes the proof.

Completion of the proof of Theorem 1.3

By Corollary 3.4, \( O = C(S(0)) \). By Lemma 3.5, \( S(0) = P \), so \( O = C(P) \) as claimed.