MATH 240: HOMEWORK #4

DUE IN FLORA'S MAILBOX BY NOON ON NOV. 25.

1. The drag equation

Suppose that an object of mass m is falling toward the earth, and that it has height $h(t)$ above the surface at time t. Newton's law says that

$$\text{Force} = \text{mass} \times \text{acceleration}. \tag{1.1}$$

Two kinds of forces act on the object:

A. A downward force of gravity having magnitude mg, where g is the gravitation constant.
B. The drag force F_D due to wind resistance. This acts in the opposite direction to the velocity $h'(t)$, and its magnitude is

$$|F_D| = cA(t)h'(t)^2 \tag{1.2}$$

where $c > 0$ is constant and $A(t)$ is the cross sectional area of the object at time t.

Notice that if $A(t)$ is constant, the drag force goes up as the square of the velocity. This is due to the fact that the energy imparted by each molecule of air hit during the fall is proportion to velocity, and the number of molecules hit per second is also proportional to velocity.

1. Show that if we assume $h(t)$ is decreasing with time (corresponding to falling toward the earth), (1.1) becomes

$$-mg + cA(t)h'(t)^2 = mh''(t) \tag{1.3}$$

In particular, explain the signs of the terms on the left.

2. Suppose now that the object is a parachutist with a circular parachute. They open the parachute so that it's radius is $b/\sqrt{|h'(t)|}$ at time t for some constant $b > 0$. Are they making the parachute larger or smaller as the velocity decreases in magnitude? What is $A(t)$ in this case, and what differental equation does (1.3) become? Note: Be careful to make sure that the signs of terms in the differential equation agree with the fact that the drag force acts in the opposite direction to the velocity of the parachutist.
3. We have not specified any of the constants in the differential equation appearing in part (2). Show that the differential equation can be determined from the initial height $h(0)$, the initial velocity $h'(0)$ and from the terminal velocity v_∞. The terminal velocity v_∞ can be defined as the number such that $H(t) = -v_\infty t$ is a solution of the differerential equation. This solution would lead to a constant velocity of $H'(t) = -v_\infty$. Explain why the terminal velocity corresponds to an exact balance between gravity and the drag force.

2. Using Wolfram alpha

When using differential equation to model physical systems, various kinds of software can be very useful. One example is Wolffram alpha. Suppose, for instance, that in problem #2 above, we try to take into account the fact that wind resistance is proportional to atmospheric density, and this density decreases exponentially with height. The drag force is thus multiplied by $e^{-rh(t)}$ for some constant $r > 0$.

4. Write down the differential equation which the one in problem #2 becomes when one multiplies the drag force by $e^{-rh(t)}$. Is this a linear differential equation?

5. Suppose for simplicity that the differential equation becomes

$$h''(t) = -5e^{-7h(t)}h'(t) - 11 \tag{2.4}$$

Do a google search for the website of Wolfram alpha, and look at the examples there about how to solve differential equations. Then use the (online) Wolfram alpha site to find an explicit form of the solution of (2.4). This involves a new function $\mathrm{erfi}(t)$. If you click on the link which gives the definition of $\mathrm{erfi}(t)$, it will tell you that $\mathrm{erfi}(0) = 0$ and the derivative of $\mathrm{erfi}(t)$ is $\mathrm{erfi}'(t) = \frac{2}{\sqrt{\pi}}e^{t^2}$. Using these facts, write down an integral giving the value of $\mathrm{erfi}(t)$ for all t.

3. Extra Credit

Suppose that instead of (2.4) one has a differential equation

$$h''(t) = -r_1 e^{-r_2 h(t)}h'(t) - r_3 \tag{3.5}$$

for some positive real constants r_1, r_2 and r_3.

6. Can you guess from problem #5 what the general solution of (3.5) is as a function of r_1, r_2 and r_3?
7. (Harder) Prove that your guess in problem #6 is correct.