MATH 280: HOMEWORK #4

DUE IN LECTURE ON FEB. 28, 2019.

1. Probability density functions associated to voter preferences

In class we've discussed the degree to which thermodynamics does and does not apply to voting preferences. This homework is about trying to ascertain when it does apply, and some consequences when it does.

Suppose that the preference of each person for a candidate ranges between 0 and 1. By a candidate we could mean a candidate for elective office, a candidate for a brand of tooth paste to buy, or any number of other possible things that people either prefer or don't. The preference of a given person for the candidate can be thought of as the probability that they will vote for the candidate over another alternative.

We define the probability density function $f(x)$ associated to the preferences of a given population for a particular candidate in the following way. We assume $f(x)$ is a continuous function, and that for all numbers $a \leq b$, the integral

$$\int_a^b f(x)dx \tag{1.1}$$

is the proportion of the population whose preference for the candidate is between a and b. Since preferences have to lie between 0 and 1, we have to have

$$\int_0^1 f(x)dx = 1 \quad \text{and} \quad f(x) = 0 \quad \text{if} \quad x > 1 \quad \text{or} \quad x < 0.$$

Since $f(x)$ is continuous and we can't have a negative proportion of the population with a preference in some range, one has $f(x) \geq 0$.

For example, if $f(x) = 1$ for all $0 \leq x \leq 1$, then (1.1) is just $b - a$. If we were to divide $[0, 1]$ into intervals of equal length, then the proportion of the population whose preference lies in each interval would be the same. In this sense, all preferences would be equally likely. If $f(x) = 2x$ for $0 \leq x \leq 1$, then the rate at which the proportion of the population prefers the candidate increases linearly with the preference level.

We now consider what the expected number of votes for the candidate would be, given the preference density function $f(x)$. Suppose the total population of voters is P. Suppose $[a, a + \delta]$ is a very short subinterval $[0, 1]$. The number of voters whose preference level lies in this interval is

$$P \cdot \int_a^{a+\delta} f(x)dx$$

Since δ is assumed to be very small, all of those voters have preference level approximately a, so their odds of voting for the candidate are approximately a. Thus this interval contributes approximately

$$Pa \cdot \int_a^{a+\delta} f(x)dx$$

votes, and this is appromately

$$P \int_a^{a+\delta} xf(x)dx.$$

Now let $\delta \to 0$ and sum over disjoint intervals covering $[0,1]$. This shows that the expected number of votes is

$$P\int_0^1 xf(x)dx = PE_f(X)$$

where $E_f(X)$ is the expectation of the random variable $X : \mathbb{R} \to \mathbb{R}$ defined by $X(x) = x$ relative to the probability density function $f(x)$. Thus

$$E_f(X) = \frac{PE(X)}{P} = \int_0^1 xf(x)dx$$

is the proportion of the total possible number P of yes votes that the candidate will actually receive. We can see this as the probability that a randomly chosen voter will vote for the candidate. In a sense, $E(X)$ measures the overall average energy of support for the candidate.

2. Maximizing the Shannon entropy of a distribution of preferences.

We now fix a number E in the interval $[0,1]$ and consider those $f(x)$ for which $E_f(X)$ is E.

Problem 1. The thermodynamic hypothesis in this context would require that if $f(x)$ and $g(x)$ are two probability density functions arising from voter preferences giving the same value $E_f(X) = E_g(X) = E$, then $f(x)$ and $g(x)$ are equally likely to occur. Show that there is no value of E in the open interval $(0,1)$ such that there is only one continuous function $f(x) \geq 0$ such that $\int_0^1 f(x)dx = 1$ and $E_f(X) = E$. If there any continuous probability density function $f(x)$ which can give $E_f(X) = 0$? (The second part of this problem shows why in probability theory, one has to consider more than continuous functions.)

Problem 2. Let us now suppose that the thermodynamic hypothesis holds for E. This is a usually a very strong hypothesis, since in the notation of Problem #1, there will in general be many f and g with $E_f(X) = E_g(X)$. In class we will describe how the calculus of variations (which is a continuous version of Lagrange multipliers) implies there are constants μ and c for which the following is true. Suppose f defines the probability density function with maximum Shannon entropy

$$-\int_0^1 f(x)\ln(f(x))dx$$

among all f with $E_f(X) = E$. Then

$$f(x) = ce^{-\mu x}.$$

In this problem, calculate c as a function of μ. (Hint: $\int_0^1 f(x)dx = 1$).

Problem 3. With the assumptions and notations of problem 2, find E as a function of μ.

Problem 4. With the notations of problem #3, what value of E corresponds to $\mu = 0$? How would you interpret this case? As $\mu \to -\infty$, what does E tend towards? As $\mu \to +\infty$ what does E tend towards? How would you interpret these facts in terms of the shape of graph of the preference function $f(x)$ associated to the thermodynamic equilibrium as E ranges between 0 and 1?