1. Conjugacy classes, class equation and $p$-groups

1. Suppose $G$ is a group with center $Z(G)$. Show that if the index of $Z(G)$ in $G$ is finite and equal to $n$, then the conjugacy class of each element $g$ of $G$ has at most $n$ elements. What is an easy example in which there are conjugacy classes having $n$ elements?

2. Write down explicitly all terms in the class equation for the symmetric group $S_4$. This involves, in particular, finding representatives for the conjugacy classes in $S_4$. You can use exercise (You can use exercise # 4 from homework # 2).

3. Show that if $p$ is a prime and $G$ is a finite group of order $p^n$ for some integer $n \geq 1$, then $G$ contains a subgroup of order $p^a$ for all integers $a$ in the range $0 \leq a \leq n$. (Hint: Use induction and the fact that $G$ has a non-trivial center.)

2. First applications of Sylow’s Theorem

4. If $G_1$ and $G_2$ are groups, the product group $G_1 \times G_2$ has underlying set $\{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ and componentwise multiplication. Thus $(g_1, g_2) \cdot (g_1', g_2') = (g_1g_1', g_2g_2')$. Suppose $G_1$ and $G_2$ are both equal to the symmetric group $S_3$. For all primes $p$, describe the $p$-Sylow subgroups of the product group $S_3 \times S_3$, and verify that the number of these is congruent to 1 mod $p$ and divides $\#(S_3 \times S_3)$.

5. Show that every group of order 200 has a normal 5-Sylow subgroup.

6. Suppose that $P$ is a normal $p$-Sylow subgroup of a group $G$ and that $G$ is a normal subgroup of another group $\Gamma$. Show that $P$ is normal in $\Gamma$. (Hint: A conjugate of $P$ by an element of $\Gamma$ is contained in $G$ since $G$ is normal.)