1. Finitely generated modules over a P. I. D.

1. Do problem 13 of section 12.1 of Dummit and Foote.

2. Let $R$ be a principal ideal domain. An $R$-module $M$ is irreducible if the only $R$ submodules of $M$ are $\{0\}$ and $M$. One says that $M$ is a torsion $R$-module if for each $m \in M$ there is a non-zero $r \in R$ such that $rm = 0$. Show that $M$ is an irreducible torsion $R$-module if and only if $M$ is isomorphic to $R/P$ for some non-zero prime ideal $P$ of $R$.

2. Rational canonical forms

3. Do problem 4 of section 12.2 of Dummit and Foote.

4. Do problem 9 of section 12.2 of Dummit and Foote, but only for the first of the three matrices listed for this problem. You should consider the entries of this matrix to be in an arbitrary field $F$. Does the answer depend on the characteristic of $F$?

(Recall that there is a unique ring homomorphism $\psi : \mathbb{Z} \to F$ since ring homomorphisms must send 1 to 1. If this homomorphism is injective, one says the characteristic of $F$ is 0. Otherwise, the kernel of $\psi$ is a non-zero prime ideal, and the characteristic of $F$ is the prime number $p$ which generates $I$.)

5. Find all similarity classes of two-by-two matrices $M$ in $\text{Mat}_2(\mathbb{Q})$ such that $M^4$ is the identity matrix $I$ but $M^2$ is not the identity matrix.

3. Jordan canonical form

6. Do problem 5 of section 12.3 of Dummit and Foote.

7. Suppose that $M$ is an $n \times n$ matrix with coefficients in a field $F$, and that the characteristic polynomial of $M$ is a product $\prod_{i=1}^{n} (x - \lambda_i)$ of linear polynomials in $F[x]$, where the $\lambda_i$ may not be distinct. The elements $\lambda_1, \ldots, \lambda_n$ of $F$ are called the eigenvalues of $M$. Use the Jordan canonical form of $M$ to show that for all integers $m \geq 1$, the eigenvalues of $M^m$ are $\lambda_1^m, \ldots, \lambda_n^m$.

8. Suppose $M$ is an $n \times n$ matrix with entries in a field $F$. Make the $F$-vector space $F^n$ of column vectors of size $n$ into a module for $F[x]$ by letting $x$ act as multiplication by $M$. Show that $M$ has Jordan canonical form equal to an $n \times n$ Jordan block $J(\lambda, n)$ with eigenvalue $\lambda$ if and only if $F^n$ above is isomorphic to $F[x]/((x - \lambda)^n)$ as an $F[x]$-module.