1. Rational roots and geometric constructions

1. Suppose \( f(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0 \) is a polynomial in \( x \) with coefficients \( a_i \) in \( \mathbf{Z} \). Show that any root of \( f(x) \) in \( \mathbf{Q} \) must lie in \( \mathbf{Z} \).

(Hint: Let \( r/s \) be a root in which \( r \) and \( s \) are coprime integers and \( s > 1 \). Write \( f(r/s) = 0 \) as 
\[
(r/s)^m = -a_{m-1}(r/s)^{m-1} - \cdots - a_0.
\]
Try to derive a contradiction after multiplying both sides of this equality by \( s^m \).

2. Do problem 5 of §13.3 of Dummit and Foote.

2. Extensions and roots.

3. Do problem 3 of §13.1 of Dummit and Foote. Note that you can use the description \( \mathbf{F}_2(\theta) \) as \( \mathbf{F}_2[x]/(x^3 + x + 1) \).


3. Splitting fields and separability

5. Do problem 2 of §13.4 of Dummit and Foote.

6. Write up the first of the approaches suggested in Dummit and Foote to problem 5 of §13.5.

4. Galois theory and extensions generated by radicals

7. Do problem 8 of §14.3 of Dummit and Foote.

8. Suppose \( E/F \) is a finite separable extension of fields. A Galois closure \( N \) of \( E \) over \( F \) is a minimal normal extension of \( F \) which contains \( E \). Then \( N/F \) is a finite Galois extension. If \( E = F(\alpha) \) for some element \( \alpha \) of \( E \), then \( N \) can be taken to be a splitting field for the irreducible polynomial of \( \alpha \) over \( F \). Given this information, what is the Galois closure \( N \) of the field \( E = \mathbf{Q}(\sqrt{1 + \sqrt{2}}) \) over \( F = \mathbf{Q} \)? In particular, what is the degree \([N : \mathbf{Q}]\)?


10. Do problem 3 of §14.6 of Dummit and Foote. (Hint: You can use the fact that if \( N/F \) is a finite extension of finite fields, then \( N/F \) is Galois and the Galois group \( \text{Gal}(N/F) \) is cyclic. In class we discussed the connection between discriminants and Galois groups for the splitting fields of cubic irreducible polynomials.)