MATH 502, PROBLEM SET 3.

DUE IN MATEI'S MAILBOX BY NOON ON MONDAY, NOV. 4

1. Isomorphism Theorems

1. Suppose that H is a normal subgroup of a group G and that the index of H in G is a prime number p. Show that for all subgroups K of G, either
   a. $K \subset H$, or
   b. $G = HK$ and $[K : K \cap H] = p$.

2. Composition series

2. Find all the composition series for the following groups, and prove that your list is complete.
   a. The dihedral group D_{10} of order 10.
   b. The noncyclic group $\{e, a, b, ab\}$ of order 4, where $a^2 = b^2 = e$ and $ab = ba$.

3. An alternate proof of the Jordan Holder Theorem

Let G be a finite group. These problems provide a slightly different approach to proving the Jordan Holder Theorem for G than the one we did in class or the one that is contained in problems # 6, #7, #9 and #10 of §3.4 of Dummit and Foote's book.

3. Show that G must have at least one composition series. For this you will need to use a composition series for a normal subgroup N of G and for the quotient group G/N to produce a composition series for G.

4. Suppose

$$\{e\} = N_0 \subset N_1 \subset \cdots \subset N_{k-1} \subset N_k = G \tag{3.1}$$

and

$$\{e\} = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = G \tag{3.2}$$

are two composition series for G. The Jordan Holder Theorem states that $k = r$ and that the (simple) composition factors $\{N_{i+1}/N_i\}_{i=0}^{k-1}$ agree with $\{M_{j+1}/M_j\}_{j=0}^{r-1}$ up to a rearrangement. Show that this is true by induction on $d = \min(k, r)$ via the following steps.

   a. Show that the result is true if $d = 0$. By induction we can now assume that it is true if $d < d_0$, and we must prove it when $d = d_0$.

   b. Show that the inclusion $N_{k-1} \to N_k = G = M_r$ gives an injection of groups

$$N_{k-1}/(N_{k-1} \cap M_{r-1}) \to M_r/M_{r-1} \tag{3.3}$$

   whose image is a normal subgroup of M_r/M_{r-1} Using that (3.2) is a composition series, show that either (3.3) is an isomorphism or $N_{k-1} \subset M_{r-1}$. If $N_{k-1} \subset M_{r-1}$ explain why we are done by induction, using the resulting surjection of simple groups

$$N_k/N_{k-1} = G/N_{k-1} \to G/M_{r-1} = M_r/M_{r-1}.$$

   (When can one simple group surject onto another?)

c. Use the argument in (b) with the N_i and M_j reversed to conclude that the theorem holds unless both of the homomorphism (3.3) and the analogous homorphism

$$M_{r-1}/(M_{r-1} \cap N_{k-1}) \to N_k/N_{k-1} \tag{3.4}$$

are isomorphisms.

d. Suppose now that both (3.3) and (3.4) are isomorphisms. Show that there are two composition series for the group $T = G/(N_{k-1} \cap M_{r-1})$ of the form

$$\{e\} \subset T_1 \subset T$$

and

$$\{e\} \subset T_1' \subset T$$

with $T_1 \cong M_r/M_{r-1} \cong T/T_1'$ and $T/T_1 \cong N_k/N_{k-1} \cong T_1'$. You can depict this graphically by a diagram with G at the top, $N_{k-1} \cap M_{r-1}$ at the bottom, and N_{k-1} and M_{r-1} at the same level in the middle. Indicate on this diagram line segments corresponding to the quotients $N_{k-1}/(N_{k-1} \cap M_{r-1})$, $N_k/N_{k-1} = G/N_{k-1}$, $M_{r-1}/(N_{k-1} \cap M_{r-1})$ and $M_r/M_{r-1} = G/M_{r-1}$. Mark pairs of lines segments which corresponds to quotient groups which must be isomorphic.

e. Continuing with the hypothesis of (d), take a composition series

$$\{e\} = L_0 \subset L_1 \subset \cdots L_{w-1} \subset L_w = L = N_{k-1} \cap M_{r-1}$$

for the group $L = N_{k-1} \cap M_{r-1}$. Show how to use this composition series together with the diagram in (d) give a composition series for N_{k-1} and also one for M_{r-1}.

f. Suppose $k = \min(k, r)$. What are the lengths of the composition series for N_{k-1} which result from step (e) and from the composition series which results from the first part of the series (3.1)? Explain why you can use induction on $\min(k, r)$ to compare these series. Having made this comparison, use it to finish the proof by comparing the composition series for M_{r-1} resulting from (3.2) and part (e). You will also need part (d) to finish the comparison of composition factors.

4. Solvable groups

5. Show that if N is a normal subgroup of a group G then G is solvable if and only if both N and G/N are solvable.
6. Show that the isometry group of $\mathbb{R}^2$ with the usual Euclidean metric is solvable. (Hint: First show that is is enough to prove $O(2)$ is solvable. Then show that $O(2)$ contains with index 2 the subgroup $SO(2)$ or rotations about the origin, and explain why $SO(2)$ is abelian.)