MATH 502, PROBLEM SET 4.

DUE IN MATEI'S MAILBOX BY NOON ON DEC. 2

1. Product rings

I will assume in what follows that all rings R have a multiplicative identity 1_R. Suppose J is a non-empty set and that $\{R_i\}_{i\in J}$ is a collection of rings indexed by J. The underlying set of the product ring $R = \prod_{i\in J} R_i$ is the set of all functions $f :\to \cup_{i\in J} R_i$ such that $f(i) \in R_i$ for all $i \in J$. (One can think of f as corresponding to a vector whose i-th component is $f(i) \in R_i$). The ring operations $+$ and $\cdot$ on R are those which result from adding and multiplying functions on J using the addition and multiplication of the R_i's.

1.1. Describe the unit group R^* of R using the unit groups of the rings R_i.

1.2. An element a of a ring R is a zero divisor if a is not 0 and there is a non-zero b in R such that either $ab = 0$ or $ba = 0$. Suppose each of the R_i has a non-zero element. Under what conditions on J and the R_i associated to $i \in J$ does R have no zero divisors?

1.3. An element b of a ring A is an idempotent if $b^2 = b$. Show that if J_0 is a subset of J, then there is an idempotent $b \in R$ defined by $b(j) = 1_{R_i}$ if $j \in J_0$ and $b(j) = 0$ if $j \notin J_0$. Prove that these are the only idempotents if each R_i has no zero-divisors.

1.4. Suppose J is finite. Show that every left ideal I of R has the form $\prod_{i\in J} I_i$, where I_i is a left ideal of I_i and $f \in R$ is in $\prod_{i\in J} I_i$ exactly when $f(i) \in I_i$ for all $i \in R$. (Hint: Use the idempotents b of problem # 3 which are associated to subsets J_0 which have a single element.)

1.5. A left ideal $\mathcal{P}$ of a ring A is proper if $\mathcal{P} \neq A$. Call $\mathcal{P}$ a maximal left ideal of it is proper and there is no proper left ideal $\mathcal{Q}$ of A which contains $\mathcal{P}$ but is not equal to $\mathcal{P}$. Show that if J is finite, then the ideals I in problem # 1.4 which are maximal are those for which there is some $i \in J$ for which I_i is a maximal left ideal in R_i and $I_j = R_j$ for $j \neq J$.

1.6. An ideal $\mathcal{P}$ in a commutative ring A is a prime ideal if it is a proper ideal with the following property. If $a, b \in A$ and $a \cdot b \in \mathcal{P}$ then either $a \in \mathcal{P}$ or $b \in \mathcal{P}$. Show that if all the R_i are commutative and J is finite, then the prime ideals I of R are those for which there is some $i \in J$ such that I_i is a prime ideal of R_i and $I_j = R_j$ for $i \neq j \in J$. Show that this leads to identifying the set $\mathrm{Spec}(R)$ of prime ideals of R with the disjoint union of the sets $\mathrm{Spec}(R_i)$ as i ranges over J.

2. Prime ideals, nilpotent elements and Zorn's Lemma

In these problems, R is a commutative ring which is not the zero ring. An element α of R is nilpotent if $\alpha^n = 0$ for some integer $n \geq 1$.

2.1. Show that the set of nilpotent elements of R forms an ideal $\mathcal{N}(R)$, which is called the nilradical of R.

Hint: You can use without proof the binomial theorem, which says that

$$(\alpha + \beta)^n = \sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta^{n-i}$$

for $\alpha, \beta \in R$.

2.2. For which integers $m > 0$ is $\mathcal{N}(R) = \{0\}$ when R is the ring $\mathbb{Z}/m$?

2.3. Show that for all commutative rings R, $\mathcal{N}(R)$ is contained in every prime ideal $\mathcal{P}$ of R.

2.4. Suppose that $f \in R$ is not nilpotent. Use Zorn's Lemma to show that there is a prime ideal $\mathcal{P}$ of R which does not contain f.

(Hints: Let $\mathcal{S}$ be the set of all ideals I of R which do not contain any element of the set $\{f^i\}_{i=1}^{\infty}$. Show that $\mathcal{S}$ is not empty using that 0 is not in $\{f^i\}_{i=1}^{\infty}$. Then show that the hypotheses of Zorn's Lemma are satisfied by $\mathcal{S}$. Finally, show that a maximal element $\mathcal{P}$ of S has to be a prime ideal. For this step, observe that if $\alpha \notin \mathcal{P}$ then the ideal $R\alpha + \mathcal{P}$ generated by α and $\mathcal{P}$ is strictly bigger than $\mathcal{P}$, so can't be in $\mathcal{S}$. Therefore $f^i = r\alpha + m$ for some $i \geq 0$ for some $r \in R$ and $m \in \mathcal{P}$. Similarly, if $\beta \notin \mathcal{P}$ then $f^j = s\alpha + m'$ for some $j \geq 1$and some $s \in R$ and $m' \in \mathcal{P}$. Now consider $f^i \cdot f^j = f^{i+j}$ to show $\alpha \cdot \beta \notin \mathcal{P}$.)

2.5. Use problems 2.3 and 2.4 to show that $\mathcal{N}(R) = \cap_{\mathcal{P}} \mathcal{P}$ where the intersection is over all the prime ideals $\mathcal{P}$ of $\mathcal{R}$.

3. Spectra of Rings

In these problems, R is a commutative ring. Recall that Spec(R) is the set of prime ideal $\mathcal{P}$ of R, with the following topology. The closed subsets of Spec(R) are those of the form

$$V(\mathcal{A}) = \{\mathcal{P} \in \text{Spec}(R) : \mathcal{A} \subset \mathcal{P}\}$$

as $\mathcal{A}$ ranges over all the ideal of R.

3.1. Suppose $\mathcal{Q} \in \text{Spec}(R)$. The closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ is defined to be the intersection of all closed subsets of Spec(R) which contain $\mathcal{Q}$. The reduction homomorphism $r : R \to R/\mathcal{Q}$ is the ring homomorphism defined by $r(t) = t + \mathcal{Q}$ for all $t \in R$. Show that there is a bijection

$$\overline{\mathcal{Q}} \to \text{Spec}(R/\mathcal{Q})$$

which sends an ideal $\mathcal{P}$ to the ideal $r(\mathcal{P})$ of $R/\mathcal{Q}$.

Hint: First show $\overline{\mathcal{Q}} = V(\mathcal{Q})$.

3.2. The induced topology of $\overline{\mathcal{Q}}$ is the one whose closed sets have the form $V(\mathcal{A}) \cap \overline{\mathcal{Q}}$ for some ideal $\mathcal{A}$ of R. Show that the bijection in problem # 6 identifies the closed subsets of the induced topology of $\overline{\mathcal{Q}}$ with the closed subsets of Spec$(R/\mathcal{Q})$. One says that the bijection in problem # 6 is a homeomorphism of the topological spaces $\overline{\mathcal{Q}}$ and Spec$(R/\mathcal{Q})$.

3.3. In class we will discuss the case in which $R = \mathbb{C}[x, y]$ for some indeterminates x and y. Show that $\mathcal{Q} = R \cdot x$ is a prime ideal, and that $R/\mathcal{Q}$ is isomorphic to $\mathbb{C}[y]$. Using the fact that every polynomial in $\mathbb{C}[y]$ factors into a product of linear factors and the two previous problems, describe explicitly the closure of $\mathcal{Q}$ in Spec(R).