MATH 502: HOMEWORK #5

DUE AT THE FINAL EXAM

1. Localization

In these problems, R is a non-zero commutative ring and S is a multiplicatively closed subset of R which contains 1. Recall that the localization $S^{-1}R$ of R at S is the set of all formal quotients $[r/s]$ in which $r \in R$, $s \in S$ and we say that $[r/s] = [r'/s']$ if there is an $s'' \in s$ such that $s''(rs' - sr') = 0$. These quotients are added and multiplied by the usual rules for adding and multiplying fractions. There is a ring homomorphism $\pi : R \to S^{-1}R$ defined by $\pi(r) = [r/1]$. The additive identity of $S^{-1}R$ is $[0/1]$.

1. Corollary 37 of §15.4 of the Dummit and Foote text shows that the kernel π is the ideal of all $r \in R$ such that $s''r = 0$ for some $r \in R$. (This is clear from the definition of $[r/1] = [0/1]$.) Use this to show that π is an isomorphism if and only if S is contained in the multiplicative group R^* of units of R.

Hint: The issue is surjectivity. Consider when $[1/s] \in S^{-1}R$ is in the image of π.

2. Given an example in which R is a finite ring and π is not an isomorphism. Then show that if R is finite, π is always surjective.

Hint: If R is finite and $s \in S$, show $s^m = s^{m'}$ for some integers $0 < m < m'$.

2. Chinese Remainder Theorem

3. Let R be the ring $\mathbb{Q}[x]$ of polynomials with rational coefficients in one variable x. Solve for $f(x) \in \mathbb{Q}[x]$ the system of congruences

$$f(x) \equiv 1 \mod \mathbb{Q}[x](x^2 + 1)$$

$$f(x) \equiv 2 \mod \mathbb{Q}[x](x^3 + 2)$$

or explain why there is no solution.

3. Euclidean rings

Suppose R is an integral domain, i.e. a commutative ring without zero divisors. A norm on R is a function $N : R \to \mathbb{Z}_{\geq 0}$ to the non-negative integers. One says that R is a Euclidean domain if there is a norm for which the following is true. For all $a, b \in R$ with $b \neq 0$, there are $q, r \in R$ such that

$$a = qb + r \quad \text{and either} \quad r = 0 \quad \text{or} \quad N(r) < N(b).$$

In the following problems, suppose N is a Euclidean norm on R.

4. Let m be a minimal element in the image $N(R)$ of R under N. Show that every non-zero element $a \in R$ for which $N(a) = m$ must be a unit. Deduce from this that if there a non-zero element $a \in R$ with $N(a) = 0$ then a is a unit.

5. When $R = (\mathbb{Z}/2)[x]$, find the g.c.d. of $f(x) = x^4 - x$ and $g(x) = (x^2 + x + 1)(x^3 + x + 1)$.

4. Extra Credit Problems.

A. Show that the subring $R_{-3} = \mathbf{Z} + \mathbf{Z}\left(\frac{1+\sqrt{-3}}{2}\right)$ of the complex numbers is a Euclidean ring.

B. Suppose A_1 and A_2 are ideals in a commutative ring R which are not necessarily co-maximal. Show that there is an exact sequence of additive groups

$$0 \longrightarrow A_1 \cap A_2 \longrightarrow R \longrightarrow R/A_1 \bigoplus R/A_2 \xrightarrow{\mu} R/(A_1 + A_2) \longrightarrow 0.$$

C. Suppose $R = \mathbb{Z}$. Can you find a similar sequence involving more terms when one considers three ideals A_1, A_2 and A_3 of R, which need not be comaximal? What happens when R is allowed to be an arbitrary principal ideal domain?.