MATH 503: HOMEWORK #4A

DUE MARCH 31 IN YING ZONG’S MAILBOX

1. Tensor products

1.1 Do exercise # 2 of §10.4 of Dummit and Foote.
1.2 Do problem # 11 of §10.4 of Dummit and Foote.
1.3 Do problem # 16 of §10.4 of Dummit and Foote.
1.4 Do problem # 26 of §10.4 of Dummit and Foote. (See proposition 21 of §10.4 for the definition of the algebra structure of \( S \otimes R[x] \).)

2. Dehn invariants.

A polyhedron \( P \) in \( \mathbb{R}^3 \) is a bounded set which is the closure of its interior, and whose boundary is the union of finitely many flat, finite-sided polygonal faces. Faces should meet only along their edges, with each edge belonging to exactly two faces. Suppose \( e \) is an edge, and that \( F_1 \) and \( F_2 \) are the faces of \( P \) which have \( e \) in common.

2.1 Pick a point \( x \) on \( e \), and let \( n_1 \) and \( n_2 \) be the outward facing normals to \( F_1 \) and \( F_2 \) at \( x \). Write down a formula for the interior dihedral angle \( \theta(e) \) between \( F_1 \) and \( F_2 \) using the angle between \( n_1 \) and \( n_2 \). (The interior dihedral angle that the angle through which one has to rotate \( F_1 \) about \( e \) in the direction of the interior of \( P \) in order to move \( F_1 \) to \( F_2 \).) This leads to a way to compute \( \theta(e) \) using dot products.

2.2 The Dehn invariant \( D(P) \) of \( P \) is the element

\[
\sum_e l(e) \otimes \theta(e)
\]

of \( \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\mathbb{Q}\pi) \), where the sum is over all the edges \( e \) of \( P \) and \( l(e) \) is the length of \( e \). Show that

\[
D(P) = D(P_1) + D(P_2)
\]

if \( P_1 \) and \( P_2 \) are the polyhedra which result from subdividing \( P \) into two pieces by cutting \( P \) with a plane. (Hint: Consider the contribution to \( D(P_1) + D(P_2) \) from (i) the new edges which are produced by this subdivision, (ii) edges \( e \) of \( P \) which lie on the subdividing plane, and (ii) edges \( e \) of \( P \) which are cut by the subdividing plane.)

2.3 Suppose \( a, b \in \mathbb{R} \) and \([b] \) is the image of \( b \) in \( \mathbb{R}/(\mathbb{Q}\pi) \). Show that if \( a \neq 0 \) and \([b] \neq 0 \), then \( a \otimes [b] \neq 0 \) in \( \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\mathbb{Q}\pi) \). (Hint: Choose a basis for \( \mathbb{R} \) as a \( \mathbb{Q} \)-vector space which contains \( a \).)

2.4 Let \( P_1 \) be the tetrahedron in \( \mathbb{R}^3 \) which has vertices at the points \((0, 0, 0), (1, 0, 0), (1, 1, 0)\) and \((1, 1, 1)\). Show that the Dehn invariant of \( P_1 \) is trivial.

2.5 Let \( P_2 \) be the tetrahedron which has vertices \((0, 0, 0), (1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\). Show that the Dehn invariant \( D(P_2) \) is

\[
3\sqrt{2} \otimes [\gamma\pi]
\]

where

\[
\gamma\pi = \arccos\left(\frac{1}{\sqrt{3}}\right).
\]
Deduce using parts (2) and (3) of this problem that $P_1$ and $P_2$ are polyhedra of equal volume which are not scissors congruent once one shows $γ$ is not rational.

2.6 Show that $ζ = e^{γπ\sqrt{-1}}$ satisfies the equation

$$ζ + ζ^{-1} = \frac{2}{\sqrt{3}}.$$

Deduce from this that

$$ζ = \frac{1 ± \sqrt{-2}}{\sqrt{3}}.$$

Show that if $γ$ is rational, then $ζ^m = 1$ for some integer $m > 0$. Deduce from this that

$$(1 + \sqrt{-2})^m$$

is real if $γ$ is rational.

2.7 Show that there is no positive integer $m'$ such that $(1 + \sqrt{-2})^{m'}$ is real. This will complete Dehn’s proof that there are polyhedra in $\mathbb{R}^3$ which have the same volume but which are not scissors congruent. (Hint: Write $(1 + \sqrt{-2})^{m'} = a(m') + b(m')\sqrt{-2}$ for some integers $a(m')$ and $b(m')$, and use induction to consider what $a(m')$ and $b(m')$ can be modulo 3.)

The following problems are optional; they’re for extra credit.

3. QUANTUM COMPUTATION AND TENSOR PRODUCTS.

The state space of a spin one-half particle is a two dimensional complex vector space

$$\mathbb{C}^2 = \mathbb{C}v_0 ⊕ \mathbb{C}v_1$$

in which $v_0$ and $v_1$ are the base states spin up and spin down. The particle can be in any combination of base states for which $|α_0|^2 + |α_1|^2 = 1$. If the particle is observed, it will be seen in state $v_i$ with probability $|α_i|^2$. Define a Hermitian inner product on $\mathbb{C}^2$ by

$$⟨av_0 + bv_1, cv_0 + dv_1⟩ = a\overline{c} + b\overline{d}.$$

This is Hermitian in the sense that

$$⟨v, v'⟩ = ⟨v', v⟩$$

for all $v, v' ∈ \mathbb{C}^2$.

3.1 Show that a $\mathbb{C}$-linear transformation sends each allowable state (3.1) to another allowable state if and only it preserves $⟨ , ⟩$ in the sense that

$$⟨F(v), F(v')⟩ = ⟨v, v'⟩$$

for all $v$ and $v'$. Show that this is equivalent to the statement that if $M$ is the matrix of $F$ relative to the basis $\{v_0, v_1\}$, then $M · M^†$ is the identity matrix, where $M^†$ is the complex conjugate of the transpose of $M$. Under these conditions, one says $F$ and $M$ are unitary.

3.2 The state space of $n$ spin one particles is the tensor product

$$S = S_1 ⊗_\mathbb{C} \cdots ⊗_\mathbb{C} S_n = (\mathbb{C}^2)^⊗n$$

of their individual state spaces $S_1, \ldots, S_n$. Show this has a basis $B$ consisting of tensors

$$b = b_1 ⊗ \cdots ⊗ b_n$$

in which each $b_i$ is one of the two base states for particle $i$. Define a Hermitian inner product on $S$ by

$$⟨\sum_b a_b b, \sum_{b'} c_{b'} b'⟩ = \sum_b a_b \overline{c_{b'}}$$

where $b$ and $b'$ run over all pairs of elements of $B$. The allowable states $v = \sum_b a_b b$ are those for which

$$⟨v, v⟩ = \sum_b |a_b|^2 = 1$$
Then \(|a_b|^2\) is the probability of observing the \(n\)-particles in the combination of base states represented by \(b\). The allowable transformations of \(S\) are the \(\mathbb{C}\)-linear maps \(F : S \to S\) which preserve \(\langle , \rangle\), i.e. the unitary transformations. Show that if \(\psi_j : S_j \to S_j\) is a unitary transformation for each \(j\), then the tensor product

\[
F = \psi_1 \otimes \cdots \otimes \psi_n
\]

is unitary for \(S\).

**Comment:** A quantum gate is an \(F\) as in (3.2) in which all but one or two of the \(\psi_j\) are the identity map. Such a gate then acts only on one or two of the spin one particles at a time. A basic problem is to build up from compositions of such gates interesting unitary transformations. Such compositions should take some initial state to a final state which when observed will be seen with high probability to be a base state from which one can read off the answer to some problem of interest, e.g. how to factor a large integer.