MATH 503: HOMEWORK #6A

DUE AT THE FINAL EXAM ON MONDAY, MAY 5

1. Quadratic chains of fields and geometric constructions

1. Suppose \( l(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0 \) is a polynomial in \( x \) with coefficients \( a_i \) in \( \mathbb{Z} \). Show that any root of \( l(x) \) in \( \mathbb{Q} \) must lie in \( \mathbb{Z} \). (Hint: Let \( r/s \) be a root in which \( r \) and \( s \) are coprime integers and \( s > 1 \). Let \( l \) be a prime dividing \( s \) and suppose \( l^m \) is the exact power of \( l \) dividing \( s \). Expand the equality \( s^m f(r/s) = 0 \) and reduce mod \( l \) to get a contradiction.)

2. The object of this exercise is to show that there is a degree 4 extension \( K \) of \( \mathbb{Q} \) which does not contain a degree 2 extension of \( \mathbb{Q} \). This shows not every extension of \( \mathbb{Q} \) which has two-power degree is a quadratic chain.

   a. Suppose \( g(y) = y^4 + py^2 + qy + r \) is a fourth degree irreducible polynomial in \( y \) with \( p, q, r \in \mathbb{Q} \). Let the complex roots of \( g(y) \) be \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \). Define

      \[
      \theta_1 = (\alpha_1 + \alpha_2) \cdot (\alpha_3 + \alpha_4) \\
      \theta_2 = (\alpha_1 + \alpha_3) \cdot (\alpha_2 + \alpha_4) \\
      \theta_3 = (\alpha_1 + \alpha_4) \cdot (\alpha_2 + \alpha_3) \\
      h(x) = (x - \theta_1) \cdot (x - \theta_2) \cdot (x - \theta_3) = x^3 + a_2x^2 + a_1x + a_0 
      \]

      where \( a_0, a_1 \) and \( a_2 \) are complex numbers. Show that \( a_2 = -2p \).

      Hint: Write down a formula for \( a_2 \) in terms of the \( \theta_i \), and then express this in terms of the \( \alpha_i \). Then use

      \[
      g(y) = \prod_{i=1}^{4} (y - \alpha_i) = y^4 + py^2 + qy + r
      \]

   b. The polynomial \( h(x) \) is called the cubic resolvent of \( g(y) \). By calculations similar to those in part (a), which you don’t have to do, one can check that

      \[
      h(x) = x^3 - 2px^2 + (p^2 - 4r)x + q^2. 
      \]

      Show that if \( h(x) \) is irreducible in \( \mathbb{Q}[x] \), then the extension \( K = \mathbb{Q}(\alpha_1) \) is a degree 4 extension of \( \mathbb{Q} \) which does not contain a quadratic extension of \( \mathbb{Q} \). (Hint: If \( K \) contains a quadratic subfield, then \( K/\mathbb{Q} \) is a quadratic chain. In class we showed that this implies \( L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) is a quadratic chain over \( \mathbb{Q} \). Consider the degree of \( L \) over \( \mathbb{Q} \) using the subfield \( \mathbb{Q}(\theta_1) \) of \( L \).)

   c. If \( g(y) = y^4 + 2y - 2 \) show that both \( g(y) \) and \( h(x) \) are irreducible. (Hint: To handle \( h(x) \), use exercise #1 above.)

3. Do problem 5 of §13.3 of Dummit and Foote.
2. Extensions and roots.

4. Do problem 3 of §13.1 of Dummit and Foote. Note that you can use the description $\mathbb{F}_2(\theta)$ as $\mathbb{F}_2[x]/(x^3 + x + 1)$.

5. Do problem 8 of §13.2 of Dummit and Foote.

3. Splitting fields and separability


7. Write up both of the approaches suggested in Dummit and Foote to problem 5 of §13.5.

4. Galois theory and extensions generated by radicals

8. Do problem 8 of §14.3 of Dummit and Foote.

9. Do problem 1 of §14.4 of Dummit and Foote. Can you describe the Galois group over $\mathbb{Q}$ of the Galois closure?


11. Do problem 3 of §14.7 of Dummit and Foote.