Recap: Abelian categories

A category $\mathcal{C}$ is abelian if it is isomorphic to a subcategory of the category of abelian groups in such a way that:

1. $\text{Mor}_\mathcal{C}(A, B)$ is an abelian group for all objects $A$ and $B$ of $\mathcal{C}$.

2. The kernels and cokernels of elements of $\text{Mor}_\mathcal{C}(A, B)$ are objects of $\mathcal{C}$.

3. $\mathcal{C}$ is closed under taking finite direct sums.

One can also give a more intrinsic definition of abelian categories.

Motivation: Generalize constructions which involve abelian groups, e.g. short and long exact sequences. Apply this to study functions on spaces which satisfy extra conditions.
Presheaves

Suppose $X$ is a topological space. Let $\mathcal{U}_X$ be the set of all open subsets of $X$. We can view $\mathcal{U}_X$ as a category, in which the morphisms are inclusion maps. A presheaf on $X$ is a contravariant functor

$$F : \mathcal{U}_X \to \mathcal{A} = \text{abelian groups}$$

such that $F(\emptyset) = \{0\}$.

Heuristic: A typical $F$ is defined by letting $F(U)$ be the set of functions from $U$ to some fixed abelian group $T$ which have some additional properties. If $U \to V$ is an inclusion of open sets, $F(V) \to F(U)$ can be thought of as the restriction of functions from $V$ to $U$. Call $F(U)$ the sections of $F$ over $U$.

Motivation: One natural question is whether $F(V) \to F(U)$ is surjective, i.e. whether every function of the specified type on $U$ can be extended to $V$. 
**Fact:** We can make the set $P_X$ of all presheaves on $X$ into an abelian category in the following way.

1. If $H : \mathcal{U}_X \to \mathcal{A}$ is another presheaf, a morphism $f : F \to H$ is a just a morphism of functors. Thus $f$ defines a functorial homomorphism of abelian groups

$$f(U) : F(U) \to H(U)$$

for all open subsets $U \subset X$. We use the abelian group structure of $H(U)$ to make the morphisms $\text{Mor}_{P_X}(F,H)$ into an abelian group.

2. Define $\text{PKer}(f)$ to be the presheaf for which

$$\text{PKer}(f)(U) = \text{Kernel}(f(U) : F(U) \to H(U))$$

and define $\text{PCoker}(f)$ by

$$\text{PKer}(f)(U) = \text{Cokernel}(f(U) : F(U) \to H(U))$$
**Comment:** One now needs to check the axioms to show the category $P_X$ of sheaves on $X$ is an abelian category. Alternately, one embed $P_X$ into the category of abelian groups via

$$F \to T(F) = \oplus_{\text{open } V \subset U} Z_{U,V}(F)$$

where the direct sum is over all open sets $V$ and $U$ of $X$ for which $V \subset U$, and

$$Z_{U,V}(F) = \{(a, \text{res}^U_V(a)) \in F(U) \bigoplus F(V)\}.$$ 

One needs to show that this defines an isomorphism between $P_X$ and a subcategory $\mathcal{C}_X$ of the category of abelian groups which is closed under taking kernels, cokernels and finite direct sums. This isomorphism should respect the abelian group structures on the sets of morphisms between objects.
To show $\mathcal{C}_X$ is closed under kernels and cokernels, suppose $h : F \to H$ is a morphism of sheaves. There is then an exact sequence in the category of presheaves

$$0 \to \text{Pker}(h) \to F \to H \to \text{PCoker}(h) \to 0$$

One needs to show that the exact sequence of abelian groups

$$0 \to T(\text{Pker}(h)) \to T(F) \to T(H) \to T(\text{PCoker}(h)) \to 0$$

is exact. This comes down to showing for $V \subset U$ the exactness of

$$0 \to Z_{U,V}(\text{Pker}(h)) \to Z_{U,V}(F) \to Z_{U,V}(H) \to Z_{U,V}(\text{PCoker}(h)) \to 0$$
To show that $F \to T(F)$ defines an isomorphism of $P_X$ with $C_X$, one needs to show $T$ defines a bijection between the objects and morphisms of $P_X$ and $C_X$. In particular, one needs to be able to reconstruct $F$ from $T(F)$. This is the reason for using $Z_{U,V}(F)$, which encodes the data of the restriction map $\text{res}^V_U : F(U) \to F(V)$. Note also that we can find $F(U)$ from $Z_{U,U}(F)$.

**Exercise:** Suppose $X$ is the unit interval $[0, 1]$ on the real line. Define $F$ and $H$ by

$$F(U) = \{\text{Continuous functions } r : U \to \mathbb{Z}\}$$

$$H(U) = \{\text{Continuous functions } r : U \to \mathbb{R}\}$$

We have a morphism of presheaves $f : F \to H$ in which

$$f(U) : F(U) \to H(U)$$

is induced by $\mathbb{Z} \subset \mathbb{R}$. What are the presheaf kernel and the presheaf cokernel of $f$?
Sheaves

Sheaves come about from adding conditions on a presheaf $F$ to make $F(U)$ resemble more closely the functions from an open set $U \subset X$ to an abelian group $T$.

For example, suppose

$$U = U_1 \coprod U_2$$

is the disjoint union of open subsets $U_1$ and $U_2$. If $F(U)$ were the abelian group of functions $f : U \to T$ from $U$ to some abelian group $T$, we would have

$$F(U) = F(U_1) \bigoplus F(U_2).$$  \hspace{1cm} (1)

In other words, to give a function $f : U \to T$ is the same as giving a function on $U_1$ and $U_2$, and one can specify these functions arbitrarily.
If $U = V_1 \cup V_2$ but $V_1 \cap V_2$ is not empty, then functions from $U$ to $T$ are determined by giving functions on $V_1$ and $V_2$ which agree on the overlap $V_1 \cap V_2$.

This amounts to an exact sequence

$$0 \to F(U) \to F(V_1) \bigoplus F(V_1)^{res_1-res_2} \to F(V_1 \cap V_2)$$

(2)

For a general presheaf $F$, these properties will not hold. So we make them part of the definition of a sheaf.
Definition A presheaf $F : \mathcal{U}_X \to \mathcal{A}$ on a topological space $X$ is a sheaf if it has the following properties:

1. (Local information determines sections). Suppose $\{V_i\}_i$ is a covering of an open set $U \subset X$ by open sets. If $s \in F(U)$ has image 0 in $F(V_i)$ for all $i$, then $s = 0$ in $F(U)$.

2. (Patching condition). Suppose conversely that we have sections $s_i \in F(V_i)$ for all $i$ which are consistent, in the sense that $s_i$ and $s_j \in F(U_j)$ have the same image in $F(U_i \cap U_j)$ for all $i, j$. Then there is an $s \in F(U)$ restricting to $s_i$ for all $i$. 
**Example:** If $T$ is an abelian group, and $F$ is defined by

$$F(U) = \text{Map}(U, T) = \text{functions from } U \text{ to } T$$

then $F$ is a sheaf.

**Example:** The presheaf $F$ for which $F(U)$ is the additive group of continuous functions from $U$ to $\mathbb{R}$ is a sheaf.

**Example:** Suppose $X = \mathbb{R}$. Define a presheaf $F$ on $X$ by $F(\emptyset) = \{0\}$ (as required) and $F(U) = \mathbb{Z}$ for all non-empty open sets $U$. Let the restriction maps $F(U) \to F(V)$ associated to $\emptyset \neq V \subset U$ the identity map. Then $F$ is not a sheaf, since

$$F(U_1 \cup U_2) \neq F(U_1) \oplus F(U_2)$$

if $U_1$ and $U_2$ are disjoint and non-empty.
Problem: How to make sheaves an abelian category?

Suppose $F$ and $H$ are sheaves, and that $f : F \rightarrow H$ is a morphism of presheaves. They $F$ and $H$ are presheaves, so we can consider the presheaf kernel $\text{PKer}(f)$ and the presheaf cokernel $\text{PCoker}(f)$.

It turns out that $\text{PKer}(f)$ will be a sheaf since $F$ and $H$ are sheaves. But $\text{PCoker}(f)$ need not be a sheaf. (An example is one of this week’s homework exercises.)
The Solution

We have to define sheaf cokernels in a different way than presheaf cokernels.

Following the example of coarse versus fine moduli problems, we look for the sheaf that is closest to $\text{PCoker}(f)$.

This leads to the following approach:

**Theorem:** Given any presheaf $T$, there is a sheaf $T^+$ which is closest to $T$ in the following sense. There is a morphism $\theta : T \to T^+$ of presheaves which is universal for morphisms to sheaves. So, if $r : T \to T'$ is another presheaf morphism from $T$ to a sheaf $T'$, then there is a unique morphism of sheaves $h : T^+ \to T'$ such that $r = h \circ \theta$.

**Corollary:** We can define a sheaf cokernel for a sheaf morphism $f : F \to H$ by

$$\text{Coker}(f) = (\text{PCoker}(f))^+$$