Noetherian and Artinian rings, continued

The main points from last class were:

1. A commutative ring $R$ is Noetherian (resp. Artinian) if it satisfies the ascending (resp. descending) chain condition on ideals. $R$ is Noetherian if and only if every $R$-ideal is finitely generated.

2. The Hilbert Basis theorem says $R[x]$ is Noetherian if $R$ is.

3. If $I$ is an ideal in a commutative Noetherian ring $R$, then $R/I$ is Noetherian. This follows from the fact that every ideal of $R/I$ is the reduction mod $I$ of an ideal for $R$, hence finitely generated.

4. From (2) and (3), we see that if $A$ is Noetherian (e.g. $A$ is a field, or $A = \mathbb{Z}$) then $A[x_1, \ldots, x_n]/I$ is Noetherian for every ideal $I$ of $A[x_1, \ldots, x_n]$. 
I’ll now describe without proofs some other results from Matsumura’s book.

**Thm:** Suppose $R$ is a commutative Noetherian ring, and suppose $f \in R$ is neither a zero divisor or a unit. Then every minimal prime ideal $P$ containing $f$ has height 1.

**Example:** Suppose $R$ is a U.F.D.. Consider the prime factorization

$$f = u\pi_1 \cdots \pi_n$$

of $f$, where $u$ is a unit and the $\pi_i$ are irreducibles. If $P$ is a prime ideal of $R$ containing $f$, then one of the $\pi_j$ has to lie in $P$. Hence $R\pi_j \subset P$ for some $j$, so if $P$ is minimal we have to have $P = R\pi_j$. The theorem says that $\{0\}$ is the only prime ideal $P'$ properly contained in $R\pi_j$. One of this week’s homework problems is to show this directly.
**Thm:** A Noetherian integral domain $R$ is a U.F.D. if and only if every prime ideal of height 1 is principal.

**Example:** An integral domain $R$ has dimension 1 if and only if every non-zero prime ideal is maximal and has height 1. (Why?).

**Corollary:** If $\dim(R) = 1$, the theorem says that $R$ is a U.F.D. if and only if every non-zero prime ideal is principal. This is not true if $\dim(R) > 1$, e.g. in $R = \mathbb{Z}[x]$, where $R \cdot 2 + R \cdot x$ is a non-principal prime ideal.
Homework Problem: Show that if $R$ is an integral domain which is not a field but which is finitely generated as an abelian group, then $\dim(R) = 1$.

Example: The ring

$$R = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z} + \mathbb{Z}\sqrt{-5}$$

has dimension 1. We showed before that

$$P = 2R + (1 + \sqrt{-5})R$$

is a non-principal prime ideal, so $R$ is not a U.F.D..
Artinian rings

The main result about these is proved in section 16.1 of Dummit and Foote:

**Thm:** A commutative ring $R$ is Artinian if and only if it is Noetherian and has dimension 0.

**Example:** If $k$ is a field, the ring of dual numbers over $k$ is $k[\epsilon]/(\epsilon^2)$. This is Artinian.

**Example:** For every non-zero integer $m$, the ring $\mathbb{Z}/\mathbb{Z}m$ is Artinian.

The proof uses module theory for rings, which we’ll postpone until the second term.
Cohen-Macaulay rings, regular rings and complete intersections

Suppose $A$ is a commutative ring.

**Def:** A sequence $x_1, \ldots, x_r$ of elements of $A$ is a regular sequence the following is true. The element $x_1$ is not a unit or a zero divisor of $A$. For $i = 2, \ldots, r$, the image of $x_i$ in the ring

$$A/(Ax_1 + \cdots + Ax_{i-1})$$

is not 0, not a unit and not a zero divisor. The length of the sequence $x_1, \ldots, x_r$ is $r$. (We say the empty sequence has length 0.)

**Warning:** Hartshorne forgets to require the non-unit condition in chapter II.8 of his book "Algebraic Geometry". A good reference is chapter XXI.4 of Lang, but this uses module theory.

**Example:** Suppose $A = \mathbb{Z}$. If $m \not\in \{0, \pm1\}$, then $x_1 = m$ defines a regular sequence of length 1.
**Example:** Suppose \( x_1, x_2 \) were a regular sequence of length 2 for \( A = \mathbb{Z} \). Then \( A/Ax_1 = \mathbb{Z}/(x_1) \) has to have an element \([x_2] \neq 0\) which is neither a unit or a zero divisor. Then \( d = \gcd(x_1, x_2) \) can't be \( \pm 1 \) (or else \([x_2]\) is a unit), so \([x_1/d] \neq 0\). But this implies \([x_2]\) is a zero divisor, since \([x_2] \cdot [x_1/d] = 0\) and \([x_1/d] \neq 0\). So there are no regular sequences in \( \mathbb{Z} \) of length 2.

**Def:** The depth of \( A \) is the maximal length of a regular sequence in \( A \), if this maximum exists; otherwise say \( A \) has infinite depth.

**Def:** Say \( A \) is globally Cohen-Macaulay if it is Noetherian, \( \dim(A) \) is finite and \( \dim(A) = \depth(A) \). Say that \( A \) is Cohen-Macaulay if each localization of \( A \) at a prime ideal is globally Cohen-Macalay

**Example:** We’ve just shown \( A = \mathbb{Z} \) is globally Cohen Macaulay of dimension 1. It is also Cohen-Macaulay.
**Homework** Show that if $F$ is a field, then $F[x]$ is globally Cohen Macaulay. (The same is true for $F[x_1, \cdots, x_n]$.)

**Definition** Suppose $A$ is a commutative ring and $\dim(A) = d$. An ideal $I$ of $A$ is a global complete intersection if it can be generated by $r = \dim(A) - \dim(A/I)$ elements. If this is true after replacing $A$ by its localization $A_P$ at any prime ideal $P$ containing $I$, and $I$ by $I_P = A_P \cdot I$, then one says $I$ is a local complete intersection.

**Example:** $I = (x_1) + (x_2)$ is a global and local complete intersection in $A = \mathbb{C}[x_1, x_2]$.

**Picture:**
**Def:** A local integral domain $A$ is regular if its maximal ideal can be generated by $\dim(A) < \infty$ elements. An arbitrary integral domain $A$ is called regular if all its localizations at prime ideals are regular.

**Example:** $A = \mathbb{Z}$ is regular, and any field is regular. If one localizes $\mathbb{Z}$ at a prime ideal $p\mathbb{Z}$, the resulting local ring $\mathbb{Z}(p)$ has maximal ideal $p\mathbb{Z}(p)$, which is generated by $1 = \dim(\mathbb{Z}(p))$ elements.

**Example:** The ring

$$A = \mathbb{C}[[x, y]]/(x^2 + y^3)$$

is a local integral domain, with maximal ideal

$$M = Ax + Ay$$

This $A$ has dimension 1, but one needs at least two elements to generate $M$, so $A$ is not regular.
Some facts (see Hartshorne, Lang and Matsumura).

**Thm:** Suppose $A$ is a local Noetherian ring.

1. If $A$ is regular it is Cohen Macaulay, and the ring $A[t_1, \ldots, t_n]$ is regular for all $n$.

2. Suppose $A$ is Cohen Macaulay. A set of elements $x_1, \ldots, x_r$ forms a regular sequence if and only if the ideal $I = (x_1, \ldots, x_r)$ is a global complete intersection for $A$, i.e. iff $r = \dim(A) - \dim(A/I)$.

3. Suppose $A$ is Cohen Macaulay and $x_1, \ldots, x_r$ is a regular sequence. Then $A/(x_1, \ldots, x_r)$ is Cohen Macaulay.

**Cor:** If $A = \mathbb{Z}$ or a field, then $A[t_1, \ldots, t_n]/(x_1, \ldots, x_r)$ is locally Cohen Macaulay for all regular sequences $x_1, \ldots, x_r \in A[t_1, \ldots, t_n]$. 