MATH 602: HOMEWORK #2

DUE IN LECTURE MONDAY, SEPT. 30, 2013

1. Crystallographic groups

1. Suppose $n \geq 1$ and that $d$ is the Euclidean metric on $\mathbb{R}^n$. Let $T_n$ (resp. $O(n)$) be the group of translations (resp. the orthogonal group) inside the isometry group $\text{Isom}(\mathbb{R}^n, d)$. Let $h : O(n) \to \text{Isom}(\mathbb{R}^n, d)/T_n$ be the isomorphism induced by the inclusion of $O(n)$ into $\text{Isom}(\mathbb{R}^n, d)$. Find all elements $g$ of $O(n)$ of order two such whenever $\sigma \in \text{Isom}(\mathbb{R}^n, d)$ has image equal to $h(g)$ in $\text{Isom}(\mathbb{R}^n, d)/T_n$, then $\sigma$ has order 2. What can you say if you replace “order two” by “order $m$” for an arbitrary integer $m \geq 2$?

2. Recall that a group $\Gamma \subset \text{Isom}(\mathbb{R}^n, d)$ is crystallographic if it has the discrete topology and if there is a compact subset $C$ of $\text{Isom}(\mathbb{R}^n, d)$ such that the projection $C \to \Gamma \setminus \mathbb{R}^n$ is surjective. Use problem #1 to show that there are exactly two isomorphism classes of such $\Gamma$ when $n = 1$, one abelian and one non-abelian. How can you describe the non-abelian isomorphism class as a semi-direct product?

2. Semi-direct products.

3. Let $F$ be a field, and let $G$ be the subgroup of upper triangular matrices in $\text{GL}_n(F)$. Show that $G$ is a semi-direct product of the group $U$ of upper triangular matrices with 1’s down the diagonal with the group $D$ of diagonal matrices. Describe the associated homomorphism from $D$ to $\text{Aut}(U)$.

4. Let $H$ and $K$ be groups and let $\phi : K \to \text{Aut}(H)$ be a homomorphism. Define $\overline{\phi} : K \to \text{Out}(H)$ be the composition of $\phi$ with the natural map from $\text{Aut}(H)$ to the outer automorphism group $\text{Out}(H) = \text{Aut}(H)/\text{Inn}(H)$. Let $G$ be the semi-direct product of $H$ with $K$ which is determined by $\phi$. Prove or disprove the statement that the isomorphism class if $G$ only depends on $H$, $K$ and $\overline{\phi}$. (Hint: What is the smallest finite group $H$ for which $\text{Out}(H)$ is not isomorphic to $\text{Aut}(H)$?

3. Automorphism groups, Jordan Holder Theorem, simple groups.

5. The quaternion group $G = \{ \pm 1, \pm i, \pm j, \pm k \}$ of order 8 has center $\{1, -1\}$ and is defined by the relations that $q = (-1) \cdot q$ and $q^2 = -1$ for $q \in \{i, j, k\}$, $i \cdot j = k$, $j \cdot k = i$, $k \cdot j = i$. Find the group $\text{Inn}(G)$ of inner automorphisms of $G$, the group $\text{Aut}(G)$ of automorphisms of $G$, and the group $\text{Out}(G) = \text{Ann}(G)/\text{Inn}(G)$ of outer automorphisms of $G$. Describe the 3 different composition series for $G$.

6. Suppose $G$ is a finite group of even order $n$, and that $n$ is not a prime number. Suppose in addition that $G$ contains a subgroup of order $m$ for each integer $m$ dividing $n$. Show that $G$ cannot be simple. (Note: The Feit-Thompson theorem implies that every non-abelian simple group has even order, so we did not actually have to assume $G$ has even order.)

7. Suppose $G$ is a finite group and that $H$ is a normal subgroup of $G$. Prove or disprove the statement that there is a composition series for $G$ which contains $H$ as one of the subgroups in the series.