1. ABELIAN CATEGORIES.

Let

\[ 0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0 \]

be an exact sequence in an abelian category \( \mathcal{C} \). Suppose there is a morphism \( s : C \rightarrow B \) such that \( \pi \circ s \) is the identity morphism \( C \rightarrow C \).

1. Show that there is a morphism \( r : B \rightarrow A \) such that \( r \circ \iota : A \rightarrow A \) is the identity morphism of \( A \). (Hint: Consider the morphism \( \text{id}_B - s \circ \pi : B \rightarrow B \).)

2. Show that \( B \) is isomorphic both to the coproduct \( A \coprod C \) and to the product \( A \times C \) in \( \mathcal{C} \). (Hint: Use \( \iota, \pi, r \) and \( s \) to show \( B \) has the right universal properties.)

3. Suppose \( F : \mathcal{C} \rightarrow \mathcal{D} \) is an additive left exact covariant functor from \( \mathcal{C} \) to another abelian category \( \mathcal{D} \). (Recall that \( F \) is additive for every pair of objects \( A \) and \( B \) of \( \mathcal{C} \), \( F \) induces a homomorphism of abelian groups from \( \text{Mor}_\mathcal{C}(A, B) \) to \( \text{Mor}_\mathcal{D}(F(A), F(B)) \).) Show that

\[ 0 \longrightarrow F(A) \xrightarrow{\iota} F(B) \xrightarrow{\pi} F(C) \longrightarrow 0 \]

is exact and that \( F(B) \) is the coproduct of \( F(A) \) and \( F(C) \) in \( \mathcal{D} \).

2. INJECTIVE RESOLUTIONS AND DERIVED FUNCTORS.

Let \( \mathcal{C} \) be the abelian category of all abelian groups.

4. We proved in class that the injective abelian groups are exactly those which are divisible. For each finitely generated abelian group \( M \) find an injective resolution of the form

\[ 0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow 0 \]

(Hint: You can use the fundamental theorem about finitely generated abelian groups, which says that every such group is isomorphic to a finite direct sum of finite or infinite cyclic groups. Note that \( \frac{\mathbb{Z}}{n\mathbb{Z}} \) is cyclic of order \( n \).)

5. Let \( N \) be a finitely generated abelian group. Show that the functor \( F_N : \mathcal{C} \rightarrow \mathcal{C} \) defined by \( F_N(M) = \text{Hom}_\mathcal{C}(N, M) \) is a left exact, additive, covariant functor.

6. With the notation of problems #5 and #6, let \( R^iF_N : \mathcal{C} \rightarrow \mathcal{C} \) be the \( i \)th right derived functor of \( F_N \). Determine the group \( R^iF_N(M) \) for all \( i \geq 0 \) and all finitely generated \( N \) and \( M \). The usual notation for \( R^iF_N(M) \) is \( \text{Ext}^i_Z(N, M) \). (Hint: Using the fact that \( \text{Hom} \) commutes over finite direct sums, reduce to the case in which \( N \) and \( M \) each have a single generator. Then use the resolutions of problem #5.)
3. **Zorn’s Lemma.**

7. Suppose $R$ is a commutative ring which is not the zero ring, i.e. such that $1_R$ is not $0_R$. An element $x \in R$ is nilpotent if $x^n = 0_R$ for some positive integer $n$. Recall that a (two-sided) ideal of $R$ is an additive subgroup $J$ of $R$ such that $rJ \subseteq J$ and $Jr \subseteq J$ for all $r \in R$.

7A. Show that the set $\text{nil}(R)$ of nilpotent elements of $R$ is an ideal. This ideal is called the nilradical of $R$.

7B. An ideal $J$ of $R$ is prime if $J$ is a proper ideal of $R$, and whenever $x$ and $y$ are elements of $R$ such that $xy \in J$, then one of $x$ or $y$ is in $J$. Show that $\text{nil}(R)$ is contained in the intersection $I$ of all the prime ideals of $R$.

7C. Show that $\text{nil}(R) = I$. (Hint: If $x$ is an element of $R$ which is not nilpotent, apply Zorn’s Lemma to the collection of ideals which contain no positive power of $x$.)

4. **Group cohomology and projective resolutions**

8. Suppose $G$ is a group. Let $R = \mathbb{Z}[G]$ be the group ring of $G$. Thus the elements of $R$ are finite integral combinations of the elements of $G$. The additive group $\mathbb{Z}$ becomes an $R$-module if we let every element of $G$ act as the identity on $\mathbb{Z}$. Thus if $\alpha = a_1 g_1 + \cdots + a_n g_n$ is an element of $R$ with $a_i \in \mathbb{Z}$ and $g_i \in G$, then

$$\alpha \cdot n = \sum_i a_i n$$

for $n \in \mathbb{Z}$. A left $R$-module $M$ is simply an abelian group with a left action of $G$. The cohomology groups $H^i(G, M)$ can be defined as in class using an injective resolution of $M$ by $R$-modules. It can be shown that $H^i(G, M)$ is also isomorphic to the group $\text{Ext}^i_R(\mathbb{Z}, M)$ defined by Dummit and Foote in the Definition following displayed equation (17.7) in section 17.1 of their book. Using that definition, do problem #9 of section 17.2 of Dummit and Foote’s book. To do this exercise, really only need to use the definition they give of $\text{Ext}^i_R(\mathbb{Z}, M)$. 