1. Dimension theory

1. Show that if \( J \) is an ideal of \( \mathbb{Z}[x] \) which contains two non-constant irreducible polynomials \( f(x) \) and \( g(x) \) which are not multiples of each other in \( \mathbb{Q}[x] \), then the ideal \( \mathbb{Z}[x]f(x) + \mathbb{Z}[x]g(x) \) must contain a non-zero integer. Use this to show that the maximal ideals \( \mathcal{M} \) of \( \mathbb{Z}[x] \) are exactly those of the form \( \mathcal{M} = \mathbb{Z}[x]p + \mathbb{Z}[x]f(x) \) in which \( p \) is a rational prime and \( f(x) \in \mathbb{Z}[x] \) is polynomial whose reduction \( \overline{f}(x) \in (\mathbb{Z}/p)[x] \) is irreducible. Then use this to give a proof that \( \mathbb{Z}[x] \) has dimension 2.

2. A ring \( A \) is (left) Noetherian if every sequences \( \{I_j\}_{j=1}^{\infty} \) of \( A \)-ideals which is ascending, in the sense that \( I_j \subset I_{j+1} \) for all \( j \), stabilizes in the sense that for some \( N \geq 1 \) we have \( I_j = I_N \) for all \( j \geq N \). In class we will discuss the fact that if \( A \) is a Noetherian commutative ring, then \( A \) is a U.F.D. if and only if every ideal of height 1 is principal. We also discussed the fact that the subring \( A = \mathbb{Z}[\sqrt{-5}] \) of \( \mathbb{C} \) generated by \( \mathbb{Z} \) and \( \sqrt{-5} \) is not a U.F.D.. Is \( \mathbb{Z}[\sqrt{-5}] \) Noetherian? If so, exhibit a height one prime which is not principal.

3. We will also discuss in class the fact that if \( A \) is a Noetherian commutative ring, and \( f \in A \) is neither a zero divisor or a unit, then every minimal element of the set of prime ideals of \( A \) which contain \( f \) has height 1. We’ll show later if \( A \) is Noetherian then so is the polynomial ring \( A[x_1, \ldots, x_n] \) in \( n \geq 1 \) commuting variables over \( A \). Find all the minimal prime ideals \( \mathcal{P} \) of \( \mathbb{Z}[x, y] \) which contain the element \( f = x(y^2 + 1) \).

2. Module theory.

4. Let \( R \) be the field of real numbers. Make the plane \( V = \mathbb{R}^2 \) into an \( R[x] \)-module by letting \( x \) act on \( V \) as the \( R \)-linear transformation \( X : V \to V \) which is rotation clockwise about the origin by \( \pi/2 \) radians. Show that \( \{0\} \) and \( V \) are the only \( R[x] \)-submodules of \( V \).

5. Suppose \( I \) is a nilpotent ideal of the ring \( R \), in the sense that \( I^n = \{0\} \) for some integer \( n > 0 \). Suppose \( \phi : M \to N \) is a homomorphism between left \( R \)-modules such that the induced homomorphism \( \overline{\phi} : M/IM \to N/IN \) is surjective. Prove that \( \phi \) is surjective.

6. Give an example of a ring \( R \) together with \( R \)-modules \( M \) and \( N \) such that there is a homomorphism of additive groups \( h : M \to N \) which is not an \( R \)-module homomorphism. Does such an example exist if \( R \) is a quotient ring of \( \mathbb{Z} \)? What if \( R = \mathbb{Q} \)? Find all the fields \( R \) for which such an example exists.

7. An \( R \)-module \( M \) is irreducible if \( M \neq \{0\} \) and if \( \{0\} \) and \( M \) are the only \( R \)-submodules of \( M \). Show that if \( M_1 \) and \( M_2 \) are irreducible \( R \)-modules then any non-zero \( R \)-module homomorphism from \( M_1 \) to \( M_2 \) is an isomorphism. What does this say about the ring \( \text{End}_R(M) \)? (This result is known as Schur’s Lemma.)