MATH 603: HOMEWORK #4

DUE IN RYAN MANION’S MAILBOX BY FRIDAY, MARCH 21, 2014

1. Primary ideals and decompositions

1. Let $\mathcal{Q}$ be a primary ideal in a commutative ring $A$. Suppose $\mathcal{B}$ and $\mathcal{C}$ are ideals of $A$, and that $\mathcal{B} \mathcal{C} \subset \mathcal{Q}$. Suppose $\mathcal{C}$ is finitely generated. Show that either $\mathcal{B} \subset \mathcal{Q}$, or there is an integer $n \geq 1$ such that $\mathcal{C}^n \subset \mathcal{Q}$.

2. Suppose $I = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_n$ is the reduced primary decomposition of a proper ideal $I$ of a commutative ring $A$. Let $S$ be a multiplicatively closed subset of $A$. Suppose $j \geq 1$ is an integer such that $\mathcal{Q}_i$ does not intersect $S$ for $1 \leq i \leq j$ but that $\mathcal{Q}_i$ intersects $S$ for $j < i \leq n$. Show that

$$S^{-1}I = (S^{-1}\mathcal{Q}_1) \cap \cdots \cap (S^{-1}\mathcal{Q}_j)$$

is a reduced primary decomposition of $S^{-1}I$ as an ideal for $S^{-1}A$.

3. Let $M$ be a module for a commutative ring $A$. The support of $M$ is defined to be

$$\text{supp}(M) = \{ \mathcal{P} \in \text{Spec}(A) : M_{\mathcal{P}} \neq 0 \}$$

where $M_{\mathcal{P}}$ is the localization of $M$ at $\mathcal{P}$. Show that if $M$ is finitely generated as an $A$ module, then

$$\text{supp}(M) = V(\text{Ann}_A(M))$$

where $\text{Ann}_A(M)$ is the annihilator of $M$, and if $J$ is an ideal of $A$,

$$V(J) = \{ \mathcal{P} \in \text{Spec}(A) : J \subset \mathcal{P} \}$$

is the closed subset of Spec($A$) associated to $J$.

2. Fitting ideals

The following problems are from section 15.1 of Dummit and Foote’s book. Suppose $M$ is a finitely generated module over the commutative ring $R$ with generators $m_1, \ldots, m_n$. The Fitting ideal $F(M)$ (of level 0) of $M$ (also called a determinant ideal) is the ideal in $R$ generated by the determinants of all $n \times n$ matrices $A = (r_{i,j})$ where $r_{i,j} \in R$ and $r_{i,1}m_1 + \cdots + r_{i,n}m_n = 0$ in $R$, so the rows of $A$ consist of the coefficients in $R$ of relations among the generators $m_i$. In doing the problems below, you can use the basic properties of determinants described in the books of Lang and of Dummit and Foote.

4. Show that the Fitting ideal of $M$ is also the ideal in $R$ generated by all the $n \times n$ minors of all $p \times n$ matrices $A = (r_{i,j})$ for $p \geq 1$ whose rows consist of the coefficients in $R$ of relations among the generators $m_i$.

5. With $A$ as in problem #4, let $A'$ be a $p \times n$ matrix obtained from $A$ by any elementary row and column operation. Show that the ideal in $R$ generated by all the $n \times n$ minors of $A$ is the same as the ideal in $R$ generated by all the $n \times n$ minors of $A'$.

6. Suppose $m_1, \ldots, m_n$ and and $m'_1, \ldots, m'_n$ are two sets of $R$-module generators for $M$. Let $F$ denote the Fitting ideal for $M$ computed using the generators $m_1, \ldots, m_n$ and let $F'$ denote the Fitting ideal for $M$ computed using the generators $m_1, \ldots, m_n, m'_1, \ldots, m'_n$.

a. Show that $m'_s = a_{s,1}m_1 + \cdots + a_{s,n}m_n$ for some $a_{s,1}, \ldots, a_{s,n} \in R$. Deduce from this that $(-a_{s,1}, \ldots, -a_{s,n}, 0, \ldots, 0, 1, 0, \ldots, 0)$ is a relation among $m_1, \ldots, m_n, m'_1, \ldots, m'_n$. 


b. Suppose \( A = (r_{i,j}) \) is an \( n \times n \) matrix whose rows are the coefficients of relations among \( m_1, \ldots, m_n \). Show that \( \det(A) = \det(A') \) where \( A' \) is an \( (n + n') \times (n + n') \) matrix whose rows are the coefficients of relations among \( m_1, \ldots, m_n, m'_1, \ldots, m'_{n'} \). Deduce that \( F \subset F' \). (Hint: Use part (a) to find a block upper triangular \( A' \) having \( A \) in the upper left block and the \( n' \times n' \) identity matrix in the lower right block.)

c. Prove that \( F' \subset F \) and conclude that \( F' = F \). (Hint: Use part (a) to produce some elementary row operations on \( (n + n') \times (n + n') \) relation matrices, and use problem # 5.)

d. Deduce from (c) that the Fitting ideal \( F = F_R(M) \) of \( M \) is an invariant of \( M \) that does not depend on the choice of generators for \( M \) used to compute it.

7. Let \( R \) be a commutative ring. All the modules in this exercise will be assumed to be finitely generated.
   a. Suppose \( M \) can be generated by \( n \) elements. Prove that \( \text{Ann}_R(M)^n \subset F_R(M) \subset \text{Ann}_R(M) \). (You can use that if \( A \) is an \( n \times n \) matrix, and \( A' \) is the cofactor transpose of \( A \), then \( A'A \) is the diagonal matrix having each diagonal entry equal to \( \det(A) \).)
   b. Show that if \( M = M_1 \oplus M_2 \) is the direct sum of two modules \( M_1 \) and \( M_2 \) then \( F_R(M) = F_R(M_1) \cdot F_R(M_2) \).
   c. Suppose \( M = (R/I_1) \times \cdots \times (R/I_m) \) for some ideals \( I_i \) of \( R \). Show that \( F_R(M) = I_1 \cdots I_m \).
   d. Suppose \( I \) is an ideal of \( R \). Show that the image of \( F_R(M) \) in \( R/I \) is \( F_{R/I}(M/IM) \).
   e. With the notations of part (d), show that \( F_R(M/IM) \subset F_R(M) + I \subset R \).
   f. Show that if \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is an exact sequence of (finitely generated) \( R \)-modules, then \( F_R(L)F_R(N) \subset F_R(M) \).