1. **Splitting fields, separability and normality.**

1. Find the splitting field $E$ of $f(x) = x^6 + x^3 + 1$ over $\mathbb{Q}$ inside $\mathbb{C}$, and determine the degree $[E : \mathbb{Q}]$.

2. Let $F$ be a field characteristic $p > 0$, and suppose that $\alpha$ is algebraic over $F$. Show that $\alpha$ is separable over $F$ if and only if $F(\alpha) = F(\alpha^p)$ for all integers $n > 0$.

3. A field $F$ is called perfect if either $\text{char}(F) = 0$ or $\text{char}(F) = p$ and the Frobenius map $\Phi : F \rightarrow F$ defined by $\Phi(\alpha) = \alpha^p$ is an isomorphism. Show that a field $F$ is perfect if and only if every algebraic extension of $F$ is separable.

4. Suppose $F$ is a field, $f(x)$ is a monic irreducible polynomial in $F[x]$ and that $K$ is a finite normal extension of $F$. Suppose that $g(x)$ and $h(x)$ are monic irreducible factors of $f(x)$ in $K[x]$. Show that there is an automorphism $\sigma$ of $K$ over $F$ such that $\sigma(g(x)) = h(x)$, where $\sigma(g(x))$ is the polynomial which results from applying $\sigma$ to the coefficients of $g(x)$. Give an example in which this is not true if $K$ is not normal over $F$.

2. **Finite fields.**

5. Do problems 22 at the end of Chapter 5 of Lang’s “Algebra”. (You can assume the Mobius inversion formula stated in problem 21 of this chapter - this formula is not difficult to prove.)

6. Do problem 23 at the end of Chapter 5 of Lang’s “Algebra”.

3. **Straightedge and compass constructions.**

7. Do problem # 2 of section 13.3 of Dummit and Foote.

1. **Galois Theory.**


11. Suppose $f(x) \in \mathbb{Q}[x]$ is an irreducible fourth-degree polynomial and that the Galois group of $f(x)$ is the alternating group $A_4$. Show that the field $\mathbb{Q}(\alpha)$ obtained by adjoining a root $\alpha$ of $f(x)$ to $\mathbb{Q}$ is a quartic extension which has no subfield $L$ which is quadratic over $\mathbb{Q}$. Conclude that one cannot construct the point $(1, \alpha)$ in $\mathbb{R}^2$ by ruler and compass. Use the theory in Dummit and Foote’s section 14.6 (or some other method) to construct an $f(x)$ with the above properties.