This problem has to do with constructing degree 8 quaternion and dihedral extensions using class field theory.

1. Suppose $H$ is a subgroup of a finite group $G$. The transfer homomorphism

$$\text{Ver}^H_G : G^{ab} \to H^{ab}$$

between the maximal abelian quotients of $G$ and $H$ is defined in the following way. Let $T$ be a set of representatives for the right cosets of $H$ in $G$, so that $H \backslash G = \{Ht : t \in T\}$. If $g \in G$ and $t \in T$, then $tg = h_{g,t}t'$ for some $t' \in T$ and $h_{g,t} \in H$. Define

$$\text{Ver}^H_G (\overline{g}) = \overline{h} \quad \text{when} \quad h = \prod_{t \in T} h_{g,t}$$

where $\overline{g}$ (resp. $\overline{h}$) is the image of $g$ in $G^{ab}$ (resp. the image of $h$ in $H^{ab}$). Show that if $H$ is cyclic of order 8 and $G$ is a dihedral (resp. quaternion) group of order 8, then $\text{Ver}^H_G$ is trivial if $G$ is dihedral, and otherwise $\text{Ver}^H_G$ is the unique non-trivial homomorphism which has kernel the image of $H$ in $G^{ab}$.

2. Let $K$ be a global field, with idele class group $C_K = J_K/K^*$. Show that all dihedral and quaternion extensions of $K$ arise from the following construction. Let $L/K$ be a quadratic normal extension, and let $\epsilon_L : C_K \to \{\pm 1\}$ be the unique surjective homomorphism corresponding to $L$ via class field theory. Write $\text{Gal}(L/K) = \{\epsilon, \sigma\}$, with $\sigma$ of order 2. Let $\mu_4 = \{\pm 1, \pm \sqrt{-1}\}$ be the group of fourth roots of unity in $\mathbb{C}^*$. A surjective homomorphism $\chi : C_L \to \mu_4$ is of dihedral (resp. quaternion) type if:

a. $\chi^\sigma = \chi^{-1}$ when $\chi^\sigma : C_L \to \mu_4$ is defined by $\chi^\sigma(j) = \chi(\sigma(j))$ for $j \in C_L$

b. The restriction $\chi|_{C_K}$ of $\chi$ to $C_K$ via the map $C_K \to C_L$ induced by including $K$ into $L$ is trivial (in the dihedral case) or the character $\epsilon_L$ (in the quaternion case).

Let $N$ be the extension of $L$ which corresponds to the kernel of $\chi$ via class field theory over $L$. Show that $N/K$ is a dihedral (resp. quaternion) extension of degree 8 if $\chi$ is of dihedral (resp. quaternion) type, and that all such extensions arise from this construction as $L$ ranges over the quadratic Galois extensions of $K$. Which pairs $(L, \chi)$ give rise to the same $N$?

3. The character $\chi : C_L = J_L/L^* \to \mu_4$ then has local components $\chi_v : L_v^* \to \mu_4$ for each place $v$ of $L$ defined by $\chi_v(j_v) = \chi(\iota_v(j_v))$ when $\iota_v : L_v^* \to C_L$ results from the inclusion of $L_v$ into $J_L$ at the place $v$ followed by the projection $J_L \to C_L/L^*$. 

a. Suppose $K$ is a number field and that $K$ and $L$ have class number 1. Show that there are exact sequences

$$(1.1) \quad 1 \to O_L^* \to \prod_v O_v^* \to C_L \to 1 \quad \text{and} \quad 1 \to O_K^* \to \prod_w O_w^* \to C_K \to 1$$

where $v$ and $w$ range over all places of $L$ and $K$, respectively, including the archimedean places. Conclude from this that to specify a finite order continuous homomorphism
$\chi : C_L \to \mathbb{C}^*$ it is necessary and sufficient to specify continuous local characters $\chi'_w : O^*_w \to \mathbb{C}^*$ which are trivial for almost all $v$ such that $\prod_v \chi'_v$ vanishes on $O^*_L$.

b. With the notations of problem (3a), what conditions on the restrictions $\chi'_v$ are equivalent to $\chi$ being of dihedral or quaternion type? (Note that by the same reasoning, the character $\epsilon : C_K \to \{\pm 1\}$ is determined by its restrictions to the multiplicative groups $O^*_w$ of all places $w$ of $K$, and that each such $O^*_w$ embeds naturally into the product of the $O^*_v$ associated to $v$ over $w$ in $L$.)

c. Suppose $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{5})$. Show that there is a quaternion character $\chi : C_L \to \mu_4$ such that the $\chi'_v = \chi_v|O^*_v$ have the following properties. The character $\chi'_v$ is trivial unless $v$ is the unique place $v_5$ over 5 or one of the two first degree places $v_{41}$ and $v'_{41}$ over 41. The order of $\chi'_v$ is 2 if $v = v_5$ and 4 if $v = v_{41}$ or $v = v'_{41}$. Finally, when we use the natural inclusion $K = \mathbb{Q} \to L$ to identify both $O_{v_{41}}$ and $O_{v'_{41}}$ with $\mathbb{Z}_{41}$, the characters $\chi'_{v_{41}}$ and $\chi'_{v'_{41}}$ are inverses of each other when we view them both as characters of $\mathbb{Z}_{41}^*$.

2. The Carlitz Module and Class Field Theory

Homework #3 of last semester included some problems about abelian extensions of $L = \mathbb{F}_p(t)$ when $p$ is a prime which are constructed using the Carlitz module. To recall this construction, let $A = \mathbb{F}_p[t]$. One has a ring homomorphism $\psi : A \to L\{\tau\}$ sending $t$ to $t + \tau$, where $L\{\tau\}$ is the twisted polynomial ring for which $\tau^p = p\tau$ for $\beta \in L$. Then $L\{\tau\}$ acts on an algebraic closure $\overline{L}$ of $L$ by letting $\beta \in L$ act by multiplication by $\beta$, and by letting $\tau$ send $\alpha \in L$ to $\tau(\alpha) = \alpha^p$. If $\pi(t) \in A$ is not 0, define the $\pi(t)$-torsion subgroup of $\overline{L}$ by

$$\mu_{\pi(t)} = \{\alpha \in \overline{L} : \psi(\pi(t))(\alpha) = 0\}$$

Suppose $\pi(t) \in A = \mathbb{F}_p[t]$ is monic of degree $d \geq 1$ in $t$. Homework # 3 of last semester showed the following facts:

1. $\mu_{\pi(t)}$ is the set of all roots of a separable polynomial of degree $p^d$, and $\mu_{\pi(t)}$ is an additive group.

2. There is an action of the ring $A/\text{Ann}(\pi(t))A$ on $\mu_{\pi(t)}$ induced by letting the class of $h(t) \in A$ send $\alpha \in \mu_{\pi(t)}$ to $\psi(h(t))(\alpha)$. This makes $\mu_{\pi(t)}$ into a free rank one module for $A/\pi(t)A$.

3. Let $L(\mu_{\pi(t)}) = N$ be the field obtained by adjoining to $L$ all elements of $\mu_{\pi(t)}$. Suppose $\pi(t)$ is a monic irreducible polynomial of degree $d$. Let $\alpha \in \mu_{\pi(t)}$ be a generator for $\mu_{\pi(t)}$ as a free rank one module for the field $A/\text{Ann}(\pi(t))$. The integral closure of $B = \mathbb{F}_p[t]$ in the field $L(\mu_{\pi(t)})$ obtained by adjoining to $L$ all elements of $\mu_{\pi(t)}$ is the ring $B[\alpha]$ generated by $B$ and $\alpha$. (The proof is analogous to showing that $\mathbb{Z}[\zeta_p]$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\zeta_p)$.)

For simplicity we now assume that as in # 3 above, $\pi(t)$ is a monic irreducible of degree $d$. Here are some problems about relating this construction to class field theory.

a. Use #2 and #3 above to show that $N$ is an abelian extension of $L$ with Galois group equal to the unit group $(A/\pi(t)A)^*$ of the ring $A/\pi(t)A$. Show that $N$ is totally ramified over the place of $L$ associated to $\pi(t)$. (Hint: One can follow the pattern of the proof that $\mathbb{Q}(\zeta_p)$ is an abelian extension of $\mathbb{Q}$ with Galois group $(\mathbb{Z}/p)^*$.)

b. Suppose that $f(t)$ is a monic irreducible polynomial in $B = \mathbb{F}_p[t]$ which is different from $\pi(t)$. Show that the place $v$ of $L$ determined by $f(t)$ is unramified in $L$. Then show that if $w$ is any place of $N$ over $L$, the Frobenius automorphism $\text{Frob}(w) \in G = \text{Gal}(N/L)$ associated to $w$ is the image of $f(t)$ in $(A/\pi(t)A)^*$ when we identify $(A/\pi(t)A)^*$ with $G$ as in part (a) above. (Hint: To see what is going on here, write down explicitly the case in which $\pi(t) = t$ and $f(t) = t - \beta$ for some non-zero $\beta \in \mathbb{F}_p$.)
c. Conclude from part (b) that $N/L$ is unramified outside the place $v_0$ of $L = \mathbb{F}_p(t)$ determined by $\pi(t)$ and the place $v_\infty$ such that $\text{ord}_{v_\infty}(g(t)) = -\deg(g(t))$ for $g(t) \in \mathbb{F}_p[t]$. In class we shows that the degree map on the ideles $J_L$ of $L$ gives an exact sequence

\begin{equation}
1 \to J_L^0 \to J_L \to \mathbb{Z} \to 0
\end{equation}

where

\begin{equation}
J_L^0 = L^* \times \left( \prod_{v \neq v_\infty} O_v^* \right) \times (1 + t^{-1}O_{v_\infty}^*)
\end{equation}

and $t^{-1}$ is a uniformizer in $O_{v_\infty}$. Using this description and part (b), write down the Artin map

\begin{equation}
\psi_{N/L} : C_L = J_L/L^* \to \text{Gal}(N/L) = (A/\pi(t)A)^*
\end{equation}

Is $N/L$ ramified over $v_\infty$? (Hint: First consider the restriction of $\psi_{N/L}$ to $J_L^0/L^*$, and use the fact that $v_0$ is totally ramified in $N$.)