1. Disjoint extensions with coprime discriminants

This problem generalizes Proposition 17 of Chapter 3 of Lang’s “Algebraic Number Theory” book.

Suppose $L$ and $N$ are two finite separable extensions of a field $F$ inside an algebraic closure $\overline{F}$ of $F$. We will say that $L$ and $N$ are disjoint over $F$ if whenever $\{l_i\}_i$ is a basis for $L$ over $F$ and $\{w_j\}_j$ is a basis for $N$ over $F$, the set $\{l_iw_j\}_{i,j}$ is a basis for the compositum $LN$ over $F$.

Let $A$ be a Noetherian subring of $F$ such that $F = \text{Frac}(A)$ and $A$ is integrally closed in $F$. If $T$ is a field such that $F \subset T \subset LN$, let $A_T$ be the integral closure of $A$ in $T$, and let $D(A_T/A) \subset A$ be the discriminant ideal of $A_T$ over $A$. We will use without further comment the fact that if $S$ is a multiplicatively closed subset of $A$, then $S^{-1}A_T$ is the integral closure of $S^{-1}A$ in $T$ and $D(S^{-1}A_T/S^{-1}A) = S^{-1}D(A_T/A)$.

We will say that $A_L$ and $A_N$ have coprime discriminants over $A$ if for each prime ideal $P$ of $A$, either

$$(A - P)^{-1}D(A_L/A) = (A - P)^{-1}A = A_P$$

or

$$(A - P)^{-1}D(A_N/A) = (A - P)^{-1}A = A_P.$$  

The object of this exercise is to show:

**Theorem 1.1.** If $L$ and $N$ are disjoint finite separable extensions of $F$, and $A_L$ and $A_N$ have coprime discriminants over $A$, then the integral closure $A_{LN}$ of $A$ in $LN$ is the subring $A_L \cdot A_N$ generated by $A_L$ and $A_N$.

1. Show the conclusion of the Theorem will follow if we show

$$(A - P)^{-1}(A_L \cdot A_N) = (A - P)^{-1}A_{LN}$$

for all primes $P$ of $A$. Explain why we can then reduce to the case in which $A$ is a local ring and either $D(A_L/A) = A$ or $D(A_N/A) = A$.

2. Suppose $A$ is a local ring and that $D(A_N/A) = A$. Recall that $D(A_N/A)$ is the $A$-ideal generated by all discriminants $D(\{w_j\}_j)$ of bases $\{w_j\}_j$ for $N$ over $F$ such that $\{w_j\}_j \subset A_N$. Show that there is one such basis $\{w_j\}_j$ which spans the same $A$-module as its dual basis $\{w^*_\ell\}_\ell$, and that $A_N$ is the direct sum $\oplus_j Aw_j$.

3. Show that if $\{w_j\}_j$ is as in problem # 2, then a basis for $LN$ as an $L$-vector space is given by $\{w_j\}_j$. Use $\{w^*_\ell\}_\ell$ and the trace from $LN$ to $L$ to show that if $\beta = \sum_j \beta_jw_j$ lies in $A_{LN}$ for some $\beta_j \in L$, then $\beta_j \in A_L$. Deduce Theorem 1.1 from this.

4. Show that if $L/F$ and $N/F$ are finite Galois extensions, then $L$ and $N$ are disjoint over $F$ if and only if $L \cap N = F$. Is this still true if we drop the assumption that $L/F$ and $N/F$ are Galois?
2. Isometry Classes of Trace Forms

Suppose $V$ is a finite dimensional vector space over a field and that
$$\langle \cdot, \cdot \rangle : V \times V \to F$$
is a non-degenerate symmetric pairing. Let $d = \dim_{\mathbb{F}}(V)$. There is a basis $\{w_i\}_{i=1}^d$ for $V$ over $F$ such that $\langle w_i, w_j \rangle = 0$ if $i \neq j$. (This is a standard result proved by induction on dimension using the orthogonal complement of the space spanned by one non-zero element of $V$.) Two pairs $(V, \langle \cdot, \cdot \rangle)$ and $(V', \langle \cdot, \cdot \rangle')$ as above are isometric if there is an $F$-isomorphism $\psi : V \to V'$ of vector spaces which carries $\langle \cdot, \cdot \rangle$ to $\langle \cdot, \cdot \rangle'$, in the sense that
$$\langle \psi(m), \psi(m_0) \rangle' = \langle m, m_0 \rangle$$for all $m, m_0 \in V$. Let 
$$d(V, \{w_1, \ldots, w_d\}, \langle \cdot, \cdot \rangle) = \det(\{\langle w_i, w_j \rangle\}_{1 \leq i, j \leq d})$$be the discriminant of the pairing $\langle \cdot, \cdot \rangle$ on $V$ relative to a basis $\{w_1, \ldots, w_d\}$ of $V$ over $F$.

5. Show that the class $h_i(V, \langle \cdot, \cdot \rangle)$ of $d(V, \{w_1, \ldots, w_d\}, \langle \cdot, \cdot \rangle)$ in the quotient group $F^*/(F^*)^2$ does not depend on the choice of $\{w_1, \ldots, w_d\}$, and is an invariant of the isometry class of $(V, \langle \cdot, \cdot \rangle)$.

6. Suppose $F$ is any field of characteristic not equal to 2. Let $L$ be a quadratic extension field of $F$, considered as an $F$-vector space. Let $\text{Tr}_{L/F} : L \times L \to F$ be the trace pairing. Show that the isometry class of $(L, \text{Tr}_{L/F})$ as a two-dimensional vector space with a quadratic form determines the quadratic extension $L/F$, in the following sense. If $L'$ is another quadratic extension of $F$ and $(L, \text{Tr}_{L/F})$ is $F$-isometric to $(L', \text{Tr}_{L'/F})$ then there is an isomorphism of fields $L \to L'$ which is the identity on $F$.

7. Suppose $F$ is a field of characteristic 2. Is the conclusion of problem #6 true for separable quadratic extensions $L$ of $F$?

Comments: The class $h_i(V, \langle \cdot, \cdot \rangle)$ is called the first Hasse-Witt invariant of $(V, \langle \cdot, \cdot \rangle)$. There is a higher Hasse Witt invariant $h_{i+1}(V, \langle \cdot, \cdot \rangle)$ for each integer $i \geq 2$. The study of these when $(V, \langle \cdot, \cdot \rangle) = (L, \text{Tr}_{L/Q})$ for a number field $L$ is an active research area. An excellent book about this is “Cohomological invariants, Witt invariants, and trace forms,” by Jean-Pierre Serre, Notes by Skip Garibaldi, Univ. Lecture Ser., 28, Cohomological invariants in Galois cohomology, 1–100, Amer. Math. Soc., Providence, RI, 2003.

3. The Carlitz Module

Let $p$ be a prime, $L = \mathbb{F}_p(t)$ and $A = \mathbb{F}_p[t]$. In class we will discuss the Carlitz module defined by the ring homomorphism $\psi : A \to L\{\tau\}$ sending $t$ to $t + \tau$, where $L\{\tau\}$ is the twisted polynomial ring for which $\tau \beta = \beta^p \tau$ for $\beta \in L$. Then $L\{\tau\}$ acts on an algebraic closure $\overline{L}$ of $L$ by letting $\beta \in L$ act by multiplication by $\beta$, and by letting $\tau$ send $\alpha \in \overline{L}$ to $\tau(\alpha) = \alpha^p$. If $\pi(t) \in A$ is not 0, define the $\pi(t)$-torsion subgroup of $\overline{L}$ by
$$\mu_{\pi(t)} = \{\alpha \in \overline{L} : \psi(\pi(t))(\alpha) = 0\}$$

8. Suppose $\pi(t) \in A = \mathbb{F}_p[t]$ is monic of degree $d \geq 1$ in $t$. Show that $\mu_{\pi(t)}$ is the set of all roots of a separable polynomial of degree $p^d$, and that $\mu_{\pi(t)}$ is an additive group.

9. With the notation of problem #5, show that there is an action of the ring $A/\pi(t)A$ on $\mu_{\pi(t)}$ induced by letting the class of $h(t) \in A$ send $\alpha \in \mu_{\pi(t)}$ to $\psi(h(t))(\alpha)$. Show that this makes $\mu_{\pi(t)}$ into a free rank one module for $A/\pi(t)A$. (To prove freeness, it may be useful to factor $\pi(t)$ into a product of powers of distinct irreducibles $r(t)$ and to consider the size of $\mu_{\pi(t)} \subseteq \mu_{\pi(t)}$.)

Comment: This fact corresponds to the statement that multiplicative group of all roots of $x^n - 1$ in $\mathbb{C}$ is a free rank 1 module for the ring $\mathbb{Z}/n$. 

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10. Suppose $\pi(t)$ is a monic irreducible polynomial of degree $d$. Let $\alpha \in \mu_{\pi(t)}$ be a generator for $\mu_{\pi(t)}$ as a free rank one module for the field $A/A\pi(t)$. Try showing that the integral closure of $B = \mathbb{F}_p[t]$ in the field $L(\mu_{\pi(t)})$ obtained by adjoining to $L$ all elements of $\mu_{\pi(t)}$ is the ring $B[\alpha]$ generated by $B$ and $\alpha$. In doing this, it may be useful to construct an analog of the proof that $\mathbb{Z}[\zeta_p]$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\zeta_p)$. 