1. Hensel’s Lemma

This problem is about a generalization of Hensel’s Lemma to polynomials in two variables. Let $K$ be a $p$-adic field with integers $O_K$ and absolute value $| | : K \to \mathbb{R}$ normalized so that $|\pi_K| = q^{-1}$ when $\pi_K$ is a uniformizer in $O_K$ and $q = \#O_K/(\pi_K O_K)$. Suppose $f_1(x, y), f_2(x, y) \in O_K[x, y]$ are polynomials in two variables over $O_K$. We then have a polynomial map $K \times K \to K \times K$ defined by

$$ (x, y) \mapsto F(x, y) = (f_1(x, y), f_2(x, y)). $$

Suppose $(x_0, y_0) \in O_K \times O_K$ has the property that $F(x_0, y_0) \in (\pi_K O_K) \times (\pi_K O_K)$. Suppose $(x_0, y_0) \in O_K \times O_K$ has the property that $F(x_0, y_0) \in (\pi_K O_K) \times (\pi_K O_K)$.

1. State and prove a generalization of the naive version of Hensel’s Lemma which will provide a sufficient condition for there to exist $(x_1, y_1) \in O_K \times O_K$ such that $F(x_1, y_1) = (0, 0)$ and $(x_1, y_1) \equiv (x_0, y_0) \mod (\pi_K O_K) \times (\pi_K O_K)$.

2. Apply your criterion in problem #1 to the case in which $(x_0, y_0) = (1, -1)$, $f_1(x, y) = x^3 + xy + \pi_K$ and $f_2(x, y) = x^2 - y^2 - \pi_K$.

3. State and prove a generalization of the sophisticated form of Hensel’s Lemma based on Newton’s iteration.

2. Extensions of absolute values

Let $p$ be a prime and let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. We will sketch in class a proof that there is a unique non-archimedean absolute value $| | : \overline{\mathbb{Q}}_p \to \mathbb{R}$ which extends the usual $p$-adic absolute value $| |_p : \mathbb{Q}_p \to \mathbb{R}$.

4. Show that if $\alpha \in \overline{\mathbb{Q}}_p$ and $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ then $|\sigma(\alpha)|_p = |\alpha|_p$.

5. Suppose that $f(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in \mathbb{Q}_p$ for some $n \geq 2$ and $b_i \in \mathbb{Z}_p$. If $b_0 \in \mathbb{Z}_p$ and $b_j \notin \mathbb{Z}_p$ for some $0 < j < n$, is it possible that $f(x)$ is irreducible in $\mathbb{Q}_p[x]$?