MATH 720: HOMEWORK #1

1. Constructing quaternion and dihedral extensions by class field theory.

This problem has to do with constructing degree 8 quaternion and dihedral extensions using class field theory.

1. Suppose H is a subgroup of finite index in a group G. The transfer homomorphism

$$\mathrm{Ver}_G^H : G^{ab} \to H^{ab}$$

between the maximal abelian quotients of G and H is defined in the following way. Let T be a set of representatives for the right cosets of H in G, so that $H\backslash G = \{Ht : t \in T\}$. If $g \in G$ and $t \in T$, then $tg = h_{g,t}t'$ for some $t' \in T$ and $h_{g,t} \in H$. Define

$$\mathrm{Ver}_G^H(\overline{g}) = \overline{h} \quad \text{when} \quad h = \prod_{t \in T} h_{g,t}$$

where $\overline{g}$ (resp. $\overline{h}$) is the image of g in G^{ab} (resp. the image of h in H^{ab}). Show that if H is cyclic of order 8 and G is a dihedral (resp. quaternion) group of order 8, then Ver_G^H is trivial if G is dihedral, and otherwise Ver_G^H is the unique non-trivial homomorphism which has kernel the image of H in G^{ab}.

2. Let L/K be a finite extension of global fields. Define $C_K = J_K/K^*$ to be the idele class group of K. Let K^{ab} be the maximal abelian extension of K in some algebraic closure containining L. Two basic properties of the Artin map $\Psi_K : C_K \to \mathrm{Gal}(K^{ab}/K)$ are that the two following two diagrams commute:

(1.1)
$$\begin{array}{ccc} C_L & \xrightarrow{\Psi_L} & \mathrm{Gal}(L^{ab}/L) \\ {\scriptstyle Norm_{L/K}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{L^{ab}/K^{ab}}} \\ C_K & \xrightarrow{\Psi_L} & \mathrm{Gal}(K^{ab}/K) \end{array}$$

(1.2)
$$\begin{array}{ccc} C_K & \xrightarrow{\Psi_L} & \mathrm{Gal}(K^{ab}/K) \\ {\scriptstyle i_{K/L}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ver}_{L/K}} \\ C_L & \xrightarrow{\Psi_K} & \mathrm{Gal}(L^{ab}/L) \end{array}$$

in which $\mathrm{res}_{L^{ab}/K^{ab}}$ is induced by restriction, $i_{K/L}$ is induced by the inclusion of K into L and $\mathrm{Ver}_{L/K}$ is the transfer map.

Use this to show that all dihedral and quaternion extensions of K arise from the following construction. Let L/K be a quadratic separable extension, and let $\epsilon_L : C_K \to \{\pm 1\}$ be the unique surjective homomorphism corresponding to L via class field theory. Write $\mathrm{Gal}(L/K) = \{e, \sigma\}$, with σ of order 2. Let $\mu_4 = \{\pm 1, \pm\sqrt{-1}\}$ be the group of fourth roots of unity in $\mathbb{C}^*$. A surjective homomorphism $\chi : C_L \to \mu_4$ is of dihedral (resp. quaternion) type if:

a. $\chi^\sigma = \chi^{-1}$ when $\chi^\sigma : C_L \to \mu_4$ is defined by $\chi^\sigma(j) = \chi(\sigma(j))$ for $j \in C_L$

b. The restriction $\chi|_{C_K}$ of χ to C_K via the map $C_K \to C_L$ induced by including K into L is trivial (in the dihedral case) or the character ϵ_L (in the quaternion case).

Let N be the extension of L which corresponds to the kernel of χ via class field theory over L. Show that N/K is a dihedral (resp. quaternion) extension of degree 8 if χ is of dihedral (resp. quaternion) type, and that all such extensions arise from this construction as L ranges over the quadratic Galois extensions of K. Which pairs (L, χ) give rise to the same N?

3. The character $\chi : C_L = J_L/L^* \to \mu_4$ then has local components $\chi_v : L_v^* \to \mu_4$ for each place v of L defined by $\chi_v(j_v) = \chi(\iota_v(j_v))$ when $\iota_v : L_v^* \to C_L$ results from the inclusion of L_v into J_L at the place v followed by the projection $J_L \to C_L/L^*$.

a. Suppose K is a number field and that K and L have class number 1. Show that there are exact sequences

$$1 \to O_L^* \to \prod_v O_v^* \to C_L \to 1 \quad \text{and} \quad 1 \to O_K^* \to \prod_w O_w^* \to C_K \to 1 \tag{1.3}$$

where v and w range over all places of L and K, respectively, including the archimedean places. Conclude from this that to specify a finite order continuous homomorphism $\chi : C_L \to \mathbb{C}^*$ it is necessary and sufficient to specify continuous local characters $\chi_v' : O_v^* \to \mathbb{C}^*$ which are trivial for almost all v such that $\prod_v \chi_v'$ vanishes on O_L^*.

b. With the notations of problem (3a), what conditions on the restrictions χ_v' are equivalent to χ being of dihedral or quaternion type? (Note that by the same reasoning, the character $\epsilon : C_K \to \{\pm 1\}$ is determined by its restrictions to the multiplicative groups O_w^* of all places w of K, and that each such O_w^* embeds naturally into the product of the O_v^* associated to v over w in L.)

c. Suppose $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{5})$. Show that there is a quaternion character $\chi : C_L \to \mu_4$ such that the $\chi_v' = \chi_v|O_v^*$ have the following properties. The character χ_v' is trivial unless v is the unique place v_5 over 5 or one of the two first degree places v_{41} and v_{41}' over 41. The order of χ_v' is 2 if $v = v_5$ and 4 if $v = v_{41}$ or $v = v_{41}'$. Finally, when we use the natural inclusion $K = \mathbb{Q} \to L$ to identify both $O_{v_{41}}$ and $O_{v_{41}'}$ with $\mathbb{Z}_{41}$, the characters $\chi_{v_{41}}'$ and $\chi_{v_{41}'}'$ are inverses of each other when we view them both as characters of $\mathbb{Z}_{41}^*$.