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1 Problems

Question 1: Let \( L \) be the line tangent to the curve
\[
\mathbf{r}'(t) = \langle t^2 + 3t + 2, e^t \cos t, \ln(t + 1) \rangle
\]
at \( t = 0 \). Find the coordinates of the point of intersection of \( L \) and the plane \( x + y + z = 8 \).

(A) (2, 1, 0)  (B) (6, e^2 \cos 2, \ln(2))  (C) (2, 0, 1)
(D) (5, 2, 1)  (E) (8, 3, 2)  (F) (0, 4, 4)

Solution Key: 2  Solution: 3

Question 2: Which vector is perpendicular to the plane containing the three points \( P(2, 1, 5), Q(-1, 3, 4), \) and \( R(3, 0, 6) \)?

(A) \( 2\mathbf{i} - \mathbf{j} + \mathbf{k} \)  (B) \( \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \)  (C) \( 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} \)
(D) \( 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \)  (E) \( \mathbf{i} + 2\mathbf{j} + \mathbf{k} \)  (F) \( 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \)

Solution Key: 2  Solution: 3
Question 3: A force \( \vec{F} = 2\hat{i} + \hat{j} - 3\hat{k} \) is applied to a spacecraft with velocity vector \( \vec{v} = 3\hat{i} - \hat{j} \). If you express \( \vec{F} = \vec{a} + \vec{b} \) as a sum of a vector \( \vec{a} \) parallel to \( \vec{v} \) and a vector \( \vec{b} \) orthogonal to \( \vec{v} \), then \( \vec{b} \) is:

(A) \( 3\hat{i} + 6\hat{j} - \hat{k} \)  
(B) \( \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k} \)  
(C) \( \frac{3}{2}\hat{i} - \frac{3}{2}\hat{j} \)

(D) \( 8\hat{i} + 4\hat{j} - \hat{k} \)  
(E) \( 12\hat{i} - \hat{j} + \hat{k} \)  
(F) \( \hat{i} + 3\hat{j} - \hat{k} \)

Solution Key: 2\[3\]  
Solution: 3\[3\]

Question 4: Which of the following points lies on the same plane with \((1, 2, 0), (2, 2, 1), (0, 1, 1)\)?

(A) \((1, 1, 1)\)  
(B) \((4, -1, 1)\)  
(C) \((4, 1, 1)\)

(D) \((1, 1, -1)\)  
(E) \((2, -1, 3)\)  
(F) \((2, 1, 3)\)

Solution Key: 2\[4\]  
Solution: 3\[4\]
Question 5: The curvature of the curve

\[ \vec{r}(t) = 2t \hat{i} + t^2 \hat{j} - \frac{1}{3} \hat{k} \]

at the point \( t = 0 \) is

(A) 0   (B) 2   (C) \(-\frac{1}{4}\)

(D) \(\frac{1}{2}\)   (E) \(-1\)   (F) none of the above

Solution Key: 2\[\text{B}\]  Solution: 3\[\text{B}\]

Question 6: Which of the following surfaces intersect the plane \( x = 2 \) at a parabola?

(A) \(-\frac{z^2}{2} = \frac{x^2}{9} + \frac{y^2}{4}\)   (B) \(\frac{z^2}{4} = \frac{x^2}{9} + \frac{y^2}{4} - 1\)

(C) \(\frac{z^2}{4} = \frac{x^2}{9} + \frac{y^2}{4}\)   (D) \(\frac{z}{2} = \frac{x^2}{9} - \frac{y^2}{4}\)

(E) \(\frac{z^2}{4} = \frac{x^2}{9} + \frac{y^2}{4} + 1\)   (F) \(-\frac{z^2}{25} = \frac{x^2}{9} + \frac{y^2}{4}\)

Solution Key: 2\[\text{B}\]  Solution: 3\[\text{B}\]
**Question 7:** The set of points $P$ such that $\overrightarrow{QP} \cdot \overrightarrow{B} = 2$ is

(A) a line though $Q$ parallel to $\overrightarrow{B}$

(B) a plane through $Q$ parallel to $\overrightarrow{B}$

(C) a plane through $Q$ perpendicular to $\overrightarrow{B}$

(D) a line parallel to $\overrightarrow{B}$ but not passing through $Q$

(E) a plane perpendicular to $\overrightarrow{B}$ but not passing through $Q$

(F) a line through $Q$ and perpendicular to $\overrightarrow{B}$

**Solution Key: 2**

Solution: 3
Question 8: The trout in a pond is harvested at a constant rate of $H$ trout per day. It is known that the growth of the trout population is governed by the logistic equation with harvesting:

$$\frac{dP}{dt} = 12P - P^2 - H. $$

(a) For what harvesting rates will this growth model have two equilibrium populations? Will these equilibria be stable or unstable?

(b) Determine the special value of $H$ for which the population growth has a single equilibrium. What will happen if we start harvesting at a rate higher than this special value?

Solution Key: 2

Solution: 3

Question 9: True or false. Explain your reasoning.

(a) The line $\vec{r}(t) = (1 + 2t)\hat{i} + (1 + 3t)\hat{j} + (1 + 4t)\hat{k}$ is perpendicular to the plane $2x + 3y - 4z = 9$.

(b) The equation $x^2 = z^2$ in three dimensions, describes an ellipsoid.

(c) $|\vec{a} \times \vec{b}| = 0$ implies that either $\vec{a} = 0$ or $\vec{b} = 0$.

Solution Key: 2

Solution: 3

Question 10: A bug is crawling along a helix and his position at time $t$ is given by $\vec{r}(t) = (\sin(2t), \cos(2t), t)$. Which of the following statements are true and which are false? Explain and justify your reasoning.

(a) The unit normal vector always points toward the $z$-axis.

(b) The bug travels upward at a constant rate, i.e. the unit tangent vector has a constant $z$-component at any moment of time.

(c) The unit binormal vector always points straight up or straight down.
2 Solution key

(1) (D)
(2) (E)
(3) (B)
(4) (F)
(5) (D)
(6) (D)
(7) (E)
(8) (a) $H < 36$, one unstable and one stable equilibrium; (b) to have one equilibrium we must harvest at a rate $H = 36$. If $H > 36$, then the trout population will go extinct.
(9) (a), (b), and (c) are false
(10) (a) is true, (b) is true, and (c) is false.
3 Solutions

Solution of problem 1.1: The point $P$ on the curve corresponding to the value of the parameter $t = 0$ has position vector $\vec{r}(0) = \langle 2, 1, 0 \rangle$. In other words $P$ has coordinates $P(2, 1, 0)$. The tangent vector to the curve at $P$ is the vector $\frac{d\vec{r}}{dt}(0) = \left( \frac{d}{dt}(t^2 + 3t + 2, e^t \cos t, \ln(t + 1)) \right)_{t=0}$

$$= \left( \langle 2t + 3, e^t \cos t - e^t \sin t, \frac{1}{t+1} \rangle \right)_{t=0}$$

$$= \langle 3, 1, 1 \rangle.$$

The tangent line $L$ at $t = 0$ is the line passing through $P$ and having the tangent vector $\frac{d\vec{r}}{dt}(0)$ as a direction vector. Thus $L$ is given by the parametric equations

$$x = 2 + 3s,$$

$$y = 1 + s,$$

$$z = s.$$

To intersect the line $L$ with the plane $x + y + z = 8$ we substitute the parametric expressions for $x$, $y$, and $z$ in the equation of the plane and solve for $s$. We get $(2 + 3s) + (1 + s) + s = 8$, i.e. $s = 1$. Thus the point of intersection is the point on the line which corresponds to the value of the parameter $s = 1$, i.e. the point $(5, 2, 1)$. The correct answer is (D).

Solution of problem 1.2: A vector is perpendicular to a plane if and only if it is parallel to a normal vector for the plane.

To find a vector $\vec{n}$ which is normal to the plane containing $P(2, 1, 5)$, $Q(-1, 3, 4)$, and $R(3, 0, 6)$ we need to find two non-parallel vectors in the plane and compute their cross products. The vectors

$$\vec{PQ} = \langle -1 - 2, 3 - 1, 4 - 5 \rangle = \langle -3, 2, -1 \rangle$$

$$\vec{PR} = \langle 3 - 2, 0 - 1, 6 - 5 \rangle = \langle 1, -1, 1 \rangle$$
are not parallel and belong to the plane, so we have

\[ \mathbf{\overrightarrow{n}} = \mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = (2 - 1)\hat{i} + (3 - 1)\hat{j} + (3 - 2)\hat{k} \]

\[ = \hat{i} + 2\hat{j} + \hat{k} \]

The correct answer is (E).

**Solution of problem 1.3:** If \( \mathbf{\overrightarrow{F}} = \mathbf{\overrightarrow{a}} + \mathbf{\overrightarrow{b}} \) with \( \mathbf{\overrightarrow{a}} \parallel \mathbf{\overrightarrow{v}} \) and \( \mathbf{\overrightarrow{b}} \perp \mathbf{\overrightarrow{v}} \), then \( \mathbf{\overrightarrow{a}} \) is the orthogonal projection of \( \mathbf{\overrightarrow{F}} \) onto \( \mathbf{\overrightarrow{v}} \). From the formula for an orthogonal projection we compute

\[ \mathbf{\overrightarrow{a}} = \text{proj}_{\mathbf{\overrightarrow{v}}} \mathbf{\overrightarrow{F}} = \frac{\mathbf{\overrightarrow{F}} \cdot \mathbf{\overrightarrow{v}}}{|\mathbf{\overrightarrow{v}}|^2} \mathbf{\overrightarrow{v}} \]

\[ = \frac{\langle 2, 1, -3 \rangle \cdot \langle 3, -1, 0 \rangle}{3^2 + (-1)^2} \langle 3, -1, 0 \rangle \]

\[ = \frac{5}{10} \langle 3, -1, 0 \rangle \]

\[ = \left\langle \frac{3}{2}, \frac{-1}{2}, 0 \right\rangle. \]

Therefore

\[ \mathbf{\overrightarrow{b}} = \mathbf{\overrightarrow{F}} - \mathbf{\overrightarrow{a}} \]

\[ = \langle 2, 1, -3 \rangle - \left\langle \frac{3}{2}, \frac{-1}{2}, 0 \right\rangle \]

\[ = \left\langle \frac{1}{2}, \frac{3}{2}, -3 \right\rangle. \]
The correct answer is (B).

Solution of problem 1.4: If we label the points on the plane as \( P_0 = (1, 2, 0) \), \( Q_0 = (2, 2, 1) \), and \( R_0 = (0, 1, 1) \), then we can easily write two vectors parallel to the plane, e.g.

\[
\vec{u} = \overrightarrow{P_0Q_0} = (2 - 1)\hat{i} + (2 - 2)\hat{j} + (1 - 0)\hat{k} = \hat{i} + \hat{k}
\]
\[
\vec{v} = \overrightarrow{P_0R_0} = (0 - 1)\hat{i} + (1 - 2)\hat{j} + (1 - 0)\hat{k} = -\hat{i} - \hat{j} + \hat{k}.
\]

The normal vector to the plane is given by the cross-product \( \vec{n} = \vec{u} \times \vec{v} \).

We compute
\[
\vec{n} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 0 & 1 \\
-1 & -1 & 1
\end{vmatrix} = \hat{i} - 2\hat{j} - \hat{k}.
\]

Thus a point \( P = (x, y, z) \) on the plane must satisfy the equation \( \vec{n} \cdot \overrightarrow{P_0P} = 0 \), which is

\[
(x - 1) - 2(y - 2) - (z - 0) = 0
\]

or simply
\[
x - 2y - z = -3.
\]

Substituting the various choices in this equation we see that the only solution is the point \((2, 1, 3)\) corresponding to answer (F).

Solution of problem 1.5: The curvature of \( \vec{r}'(t) \) is given by

\[
\kappa(t) = \frac{|\frac{d\vec{T}}{dt}|}{|\vec{r}'|},
\]

where \( \vec{T} \) is the unit tangent vector. We compute

\[
\vec{r}''(t) = (2, 2t, 0),
\]

and so \( |\vec{r}''| = \sqrt{4 + 4t^2} \). In particular we have

\[
\vec{T} = \left< 2(4 + 4t^2)^{-\frac{1}{2}}, 2t(4 + 4t^2)^{-\frac{1}{2}}, 0 \right>.
\]
Substituting in the formula for the curvature we get

\[ \kappa(t) = \frac{\left| \frac{d}{dt} \left( 2(4 + 4t^2)^{-\frac{1}{2}}, 2t(4 + 4t^2)^{-\frac{1}{2}}, 0 \right) \right|}{(4 + 4t^2)^{\frac{3}{2}}} \]

\[ = \frac{\left| -8t(4 + 4t^2)^{-\frac{3}{2}}, 2(4 + 4t^2)^{-\frac{1}{2}} - 8t^2(4 + 4t^2)^{-\frac{3}{2}}, 0 \right|}{(4 + 4t^2)^{\frac{3}{2}}} \]

Evaluating at \( t = 0 \) we get

\[ \kappa(0) = \frac{|\langle 0, 1, 0 \rangle|}{2} = \frac{1}{2}. \]

The correct answer is (D).

\[ \square \]

**Solution of problem 1.6:** The curve of intersection of each surface with the plane \( x = 2 \) will be given by the equation in the variables \( y \) and \( z \) that is obtained from the equation of the surface after the substitution \( x = 2 \).

Substituting \( x = 2 \) in the equation of each surface we get the following equations in \( y \) and \( z \):

(A) Setting \( x = 2 \) in \( -\frac{z^2}{2} = \frac{x^2}{9} + \frac{y^2}{4} \) gives

\[ -\frac{z^2}{2} = \frac{4}{9} + \frac{y^2}{4} \]

or equivalently

\[ \frac{y^2}{4} + \frac{z^2}{2} = -\frac{4}{9}. \]

This equation has no solution so it describes the empty set. In other words the surface (A) and the plane \( x = 2 \) do not intersect.
(B) Setting \( x = 2 \) in \( \frac{z^2}{4} = \frac{x^2}{9} + \frac{y^2}{4} - 1 \) gives
\[
\frac{z^2}{4} = \frac{4}{9} + \frac{y^2}{4} - 1
\]
or equivalently
\[
\frac{y^2}{4} - \frac{z^2}{4} = \frac{5}{9}
\]
which is a scaling of a standard equation of a hyperbola.

(C) Setting \( x = 2 \) in \( \frac{z^2}{4} = \frac{x^2}{9} + \frac{y^2}{4} \) gives
\[
\frac{z^2}{4} = \frac{4}{9} + \frac{y^2}{4}
\]
or
\[
\frac{y^2}{4} - \frac{z^2}{4} = \frac{4}{9}
\]
which is again a scaling of a standard equation of a hyperbola.

(D) Setting \( x = 2 \) in \( \frac{z}{2} = \frac{x^2}{9} - \frac{y^2}{4} \) gives
\[
\frac{z}{2} = \frac{4}{9} - \frac{y^2}{4}
\]
or
\[
z = \frac{8}{9} - \frac{y^2}{2}
\]
which is the equation of a parabola.

(E) Setting \( x = 2 \) in \( \frac{z^2}{4} = \frac{x^2}{9} + \frac{y^2}{4} + 1 \) gives
\[
\frac{z^2}{4} = \frac{4}{9} + \frac{y^2}{4} + 1
\]
or
\[
\frac{z^2}{4} - \frac{y^2}{4} = \frac{13}{9}
\]
which is again a scaling of a standard equation of a hyperbola.
The correct answer is (D).

---

**Solution of problem 1.7:** Given a point $Q$ and a vector $\overrightarrow{B}$ the vector equation

$$\overrightarrow{QP} \cdot \overrightarrow{B} = 0$$

is the standard equation of the plane $\alpha$ that passes through $Q$ and is perpendicular to $\overrightarrow{Q}$. The equation $\overrightarrow{QP} \cdot \overrightarrow{B} = 2$ will describe a plane $\beta$ which is parallel to $\alpha$ and thus perpendicular to $\overrightarrow{B}$. Since the right hand side in this equation is $2 \neq 0$ we conclude that $\beta$ does not pass through $Q$.

More explicitly, if $Q = (x_0, y_0, z_0)$ is a fixed point and $\overrightarrow{B} = a\hat{i} + b\hat{j} + c\hat{k}$ is a given vector, then a point $P = (x, y, z)$ satisfies the vector equation $\overrightarrow{QP} \cdot \overrightarrow{B} = 2$ if and only if the variables $x$, $y$ and $z$ satisfy the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 2.$$

Since the coefficients of $x$, $y$ and $z$ in this equation are $a$, $b$ and $c$, this is an equation of a plane with normal vector equal to $\overrightarrow{B}$. Moreover, since the right hand side of the equation is not zero, the point $Q$ can not lie on this plane. Therefore the correct choice is (E).

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**Solution of problem 1.8:** (a) The equilibria for this model are solutions of the quadratic equation $12P - P^2 - H = 0$ or equivalently $P^2 - 12P + H = 0$. The discriminant of this equation is

$$(-12)^2 - 4H = 144 - 4H = 4(36 - H).$$

Therefore the quadratic equation will have two solutions only when $H < 36$.

Suppose $H < 36$. From the quadratic formula we see that the equilibria are given by

$$P_1 = 6 - \sqrt{36 - H} \quad \text{and} \quad P_2 = 6 + \sqrt{36 - H}$$
and so
\[ \frac{dP}{dt} = -(P - P_1)(P - P_2). \]
This implies that \( dP/dt < 0 \) for \( P < P_1 \) and \( P > P_2 \) and \( dP/dt > 0 \) for \( P_1 < P < P_2 \). In particular, \( P \) is decreasing when \( P < P_1 \), \( P \) increases when \( P_1 < P < P_2 \), and \( P \) decreases again when \( P > P_2 \). This shows that both \( P = P_1 \) is an unstable equilibrium and \( P = P_2 \) is a stable equilibrium.

(b) In order to have a single equilibrium, we must choose \( H \) so that the quadratic equation \(-P^2 + 12P - H = 0\) has a unique solution. This means that the discriminant of this equation ought to be equal to zero, i.e. we ought have \( 4(36 - H) = 0 \). Thus the special value of \( H \) is \( H = 36 \). In this case, the differential equation becomes
\[ \frac{dP}{dt} = 12P - P^2 - 36 = -(P - 6)^2. \]
The unique equilibrium is at \( P = 6 \).
If \( H > 36 \) the equation has no equilibria, and moreover we have that
\[ \frac{dP}{dt} = 12P - P^2 - H = -(P - 6)^2 + (36 - H) \]
is always negative. This shows that if we harvest the trout at rate higher than the critical harvesting rate \( H = 36 \), then the population will steadily decrease and we will eventually empty the pond completely.
In contrast, if we harvest at a rate smaller than the critical harvesting rate \( H = 36 \) and if we start with enough trout in the pond, then we will always have enough fish to harvest.

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**Solution of problem 1.9:** (a) From the parametric equation \( \vec{r}(t) = (1 + 2t)\hat{i} + (1 + 3t)\hat{j} + (1 + 4t)\hat{k} \) we can extract a direction vector for the line. It is the vector \( \vec{v} \) whose components are given by the coefficients of the parameter \( t \) in the parametric equation. Thus \( \vec{v} = 2\hat{i} + 3\hat{j} + 4\hat{k} \).
Also, from the equation \( 2x + 3y - 4z = 9 \) of the plane we can extract a normal vector \( \vec{n} \) to the plane. It is the vector whose components are the coefficients of the equation of the plane. Thus \( \vec{n} = 2\hat{i} + 3\hat{j} - 4\hat{k} \).
The line and the plane will be perpendicular when \( \vec{v} \) is parallel to \( \vec{n} \), that is when \( \vec{v} \) is proportional to \( \vec{n} \). But if \( \vec{v} = c \cdot \vec{n} \) for some constant \( c \), then we will have \( 2 = 2c \), \( 3 = 3c \), and \( 4 = -4c \). From the first equation we get \( c = 1 \) but from the last equation we have \( c = -1 \). This is a contradiction. Therefore the line and the plane are not perpendicular, and so (a) is False.

(b) The equation \( x^2 = z^2 \) depends only on two variables so it describes a cylinder with a base in the \( xz \)-plane. Hence \( x^2 = z^2 \) can not be an ellipsoid and (b) is False.

(c) If the length of \( \vec{a} \times \vec{b} \) is zero, then the vector \( \vec{a} \times \vec{b} \) must be the zero vector. This can happen either when one of \( \vec{a} \) or \( \vec{b} \) is the zero vector, or when \( \vec{a} \) is parallel to \( \vec{b} \). For instance if \( \vec{a} \) is any vector and \( \vec{b} = \vec{a} \) we will have \( \vec{a} \times \vec{b} = \vec{a} \times \vec{a} = \vec{0} \). Hence (c) is False.

**Solution of problem 1.10:**

(a) The unit normal vector is given by

\[
\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left| \frac{d\vec{T}}{dt} \right|}
\]

where \( \vec{T} \) is the unit tangent vector. To compute \( \vec{T} \) we compute the velocity vector \( \frac{d\vec{r}}{dt} = \langle 2 \cos(2t), -2 \sin(2t), 1 \rangle \) and normalize

\[
\vec{T} = \frac{1}{\left| \frac{d\vec{r}}{dt} \right|} \frac{d\vec{r}}{dt}
\]

\[
= \frac{1}{\sqrt{4 \cos^2(2t) + 4 \sin^2(2t) + 1}} \langle 2 \cos(2t), -2 \sin(2t), 1 \rangle
\]

\[
= \left\langle \frac{2}{3} \cos(2t), -\frac{2}{3} \sin(2t), \frac{1}{3} \right\rangle.
\]

Hence

\[
\frac{d\vec{T}}{dt} = \left\langle -\frac{4}{3} \sin(2t), -\frac{4}{3} \cos(2t), 0 \right\rangle,
\]

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and so

\[ \vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = \frac{3}{4} \left\langle -\frac{4}{3} \sin(2t), -\frac{4}{3} \cos(2t), 0 \right\rangle = \langle -\sin(2t), -\cos(2t), 0 \rangle. \]

This is a horizontal vector pointing radially towards the z-axis. So (a) is true.

(b) From the formula for \( \vec{T} \) above we see that the z component of \( \vec{T} \) is constant and equal to 1/3. So (b) is true.

(c) The unit binormal vector is given by

\[ \vec{B} = \vec{T} \times \vec{N} \]

\[ = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{3} \cos(2t) & -\frac{2}{3} \sin(2t) & \frac{1}{3} \\ -\sin(2t) & -\cos(2t) & 0 \end{pmatrix} \]

\[ = \left\langle \frac{1}{3} \cos(2t), -\frac{1}{3} \sin(2t), -\frac{2}{3} \right\rangle. \]

This vector has non-trivial x and y components so (c) is false.