Solutions to First Exam, Math 240, Fall 2012

Question 1. Let

\[ A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \]

Find the sum of the entries in the last column of the matrix \( A^3 - 2A^2 + A \).

(A) -4 (B) 0 (C) 1

(D) -1 (E) 2 (F) 3

Answer 1. We compute

\[ A^2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 0 \\ 3 & 4 & 1 \end{bmatrix}, \]

and

\[ A^3 = \begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 0 \\ 3 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 12 & 0 \\ 0 & 8 & 0 \\ 7 & 11 & 1 \end{bmatrix}. \]

Thus for the last column of \( A^3 - 2A^2 + A \) we get

\[ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \]

So the sum of the entries in the last column of \( A^3 - 2A^2 + A \) is 0 + 0 + 0 = 0. The correct choice is (B). \( \square \)

Question 2. Consider the matrix

\[ A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 4 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \]
Which one of the following statements is **false**?

(A) The homogeneous system $Ax = 0$ has a non-zero solution.

(B) $Ax = 0$ has more than one pivot variables.

(C) If $x$ is a solution of $Ax = 0$, then we must have $x_4 = 0$.

(D) $Ax = 0$ has infinitely many solutions.

(E) $Ax = 0$ has only one free variable.

(F) $Ax = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 4 \end{pmatrix}$ has a solution.

**Answer 2.** To decide which statement is false we need to solve the system. We use elementary row operations to find the reduced row echelon form of the augmented matrix $A\#$ for the system in (F):

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
4 & 0 & 1 & 2 & 0 \\
2 & 1 & 1 & 1 & 0 \\
0 & 0 & -1 & 2 & 4
\end{bmatrix}
\]

1. $R_2 \rightarrow R_2 - 4R_1$
2. $R_3 \rightarrow R_3 - 2R_1$
3. $R_2 \leftrightarrow R_3$
4. $R_4 \rightarrow R_4 + R_3$
5. $R_2 \rightarrow R_2 - R_3$

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & -1 & -2 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Therefore $x_1, x_2, x_3$ are pivot variables, $x_4$ is a free variable, and
• The solutions to the homogeneous system $Ax = 0$ are given by

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} =
\begin{bmatrix}
  -x_4 \\
  -x_4 \\
  2x_4 \\
  x_4
\end{bmatrix}
$$

where $x_4$ can take any value.

• The solutions to the system $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}$ are given by

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} =
\begin{bmatrix}
  -x_4 + 1 \\
  -x_4 + 2 \\
  2x_4 - 4 \\
  x_4
\end{bmatrix}
$$

where again $x_4$ can take any value.

This shows that the homogeneous system has infinitely many solutions and in particular a non-zero solution, e.g. the solution corresponding to $x_4 = 1$. Similarly the system (F) has infinitely many solutions. Thus (A), (B), (D), (E), and (F) are true, and (C) is false. The correct answer is (C).

\[ \Box \]

**Question 3.** For which value of $c$ will the matrix

$$
\begin{bmatrix}
  1 & c & -1 & 2 \\
  2 & -1 & c & 5 \\
  1 & 10 & -6 & 1
\end{bmatrix}
$$

have rank equal to 2?

- (A) 1  
- (B) 2  
- (C) 3  
- (D) $-5$  
- (E) 0  
- (F) 11

**Answer 3.** To determine the rank we use ERO to bring the matrix into a row echelon form. First we can use the 1 in the upper left corner as a pivot and clear the rest of the entries in the first column by using this pivot:

$$
\begin{bmatrix}
  1 & c & -1 & 2 \\
  2 & -1 & c & 5 \\
  1 & 10 & -6 & 1
\end{bmatrix}
\quad\rightarrow\quad
\begin{bmatrix}
  1 & c & -1 & 2 \\
  0 & -1 - 2c & c + 2 & 1 \\
  0 & 10 - c & -5 & -1
\end{bmatrix}
$$

How we can proceed from here will depend on whether or not the entry \((-1 - 2c)\) is a pivot. Thus we have to consider two cases.

**Case 1.** \((-1 - 2c) \neq 0\). In this case \((-1 - 2c)\) is a pivot and we can use it to clear the entry below it:

\[
\begin{bmatrix}
1 & c & -1 & 2 \\
0 & -1 - 2c & c + 2 & 1 \\
0 & 10 - c & -5 & -1
\end{bmatrix}
\]

\[
R_2 \mapsto \frac{1}{-1 - 2c} R_2
\]

\[
\begin{bmatrix}
1 & c & -1 & 2 \\
0 & 1 & \frac{c+2}{1+2c} & -\frac{1}{1+2c} \\
0 & 10 - c & -5 & -1
\end{bmatrix}
\]

\[
R_3 \mapsto R_3 - (10 - c) R_2
\]

\[
\begin{bmatrix}
1 & c & -1 & 2 \\
0 & 1 & -\frac{c+2}{1+2c} & -\frac{1}{1+2c} \\
0 & 0 & \left(-5 + \frac{(c+2)(10-c)}{1+2c}\right) & \left(-1 + \frac{10-c}{1+2c}\right)
\end{bmatrix}
\]

For the matrix to have rank 2 this must be a row echelon form and the third row must have all of its entries equal to zero. To have the last entry of the third row equal to zero we must have

\[-1 + \frac{10 - c}{1 + 2c} = 0,
\]

or

\[1 + 2c = 10 - c,
\]

which gives \(c = 3\). Substituting this in the third entry \(\left(-5 + \frac{(c+2)(10-c)}{1+2c}\right)\) we get

\[-5 + \frac{(3+2)(10-3)}{1+2 \cdot 3} = -5 + 5 = 0.
\]

Thus \(c = 3\) gives a matrix of rank 2. For completeness we can check also

**Case 2.** \((-1 - 2c) = 0\) or \(c = -1/2\). Substituting this in the entries of the matrix

\[
\begin{bmatrix}
1 & c & -1 & 2 \\
0 & -1 - 2c & c + 2 & 1 \\
0 & 10 - c & -5 & -1
\end{bmatrix}
\]

we get

\[
\begin{bmatrix}
1 & \frac{-1}{2} & -1 & 2 \\
0 & 0 & \frac{3}{2} & 1 \\
0 & \frac{21}{2} & -5 & -1
\end{bmatrix}
\]
which is manifestly of rank 3.

Therefore to have rank 2 we must take $c = 3$ and so the correct choice is (C). \square

**Question 4.** Let $S$ be the surface parametrized by
\[
\vec{X}(s, t) = \langle t^2 + s, \, 2e^{t+s}, \, 3e^{t^2} \rangle.
\]

Let $\vec{N}(s, t)$ be the normal vector to $S$ corresponding to this parametrization. What is the magnitude of $\vec{N}$ at the point $(s, t) = (0, 0)$?

(A) 1  (B) $\sqrt{2}$  (C) $2\sqrt{3}$

(D) 5  (E) 0  (F) 2

**Answer 4.** By definition we have
\[
\vec{N}(s, t) = \frac{\partial \vec{X}(s, t)}{\partial s} \times \frac{\partial \vec{X}(s, t)}{\partial t}.
\]

For the partial derivative of $\vec{X}$ with respect to $s$ we compute
\[
\frac{\partial \vec{X}(s, t)}{\partial s} = \left\langle \frac{\partial (t^2 + s)}{\partial s}, \, \frac{\partial (2e^{t+s})}{\partial s}, \, \frac{\partial (3e^{t^2})}{\partial s} \right\rangle = \langle 1, \, 2e^{t+s}, \, 0 \rangle.
\]

Evaluating at $(s, t) = (0, 0)$ gives
\[
\frac{\partial \vec{X}}{\partial s}(0, 0) = \langle 1, \, 2, \, 0 \rangle.
\]

Similarly for the partial derivative of $\vec{X}$ with respect to $t$ we compute
\[
\frac{\partial \vec{X}(s, t)}{\partial t} = \left\langle \frac{\partial (t^2 + s)}{\partial t}, \, \frac{\partial (2e^{t+s})}{\partial t}, \, \frac{\partial (3e^{t^2})}{\partial t} \right\rangle = \langle 2t, \, 2e^{t+s}, \, 6te^{t^2} \rangle.
\]

Evaluating at $(s, t) = (0, 0)$ gives
\[
\frac{\partial \vec{X}}{\partial t}(0, 0) = \langle 0, \, 2, \, 0 \rangle.
\]
Thus  
\[ \overrightarrow{N}(0, 0) = (\hat{i} + 2\hat{j}) \times (2\hat{j}) = 2\hat{i} \times \hat{j} = 2\hat{k}. \]
In particular \(|\overrightarrow{N}(0, 0)| = |2\hat{k}| = \sqrt{0^2 + 0^2 + 2^2} = 2.\) The correct answer is (F).

**Question 5.** Let \( S \) be the closed surface consisting of the part of the paraboloid \( z = 9 - x^2 - y^2 \) lying above the \( xy \)-plane and of the disc \( x^2 + y^2 \leq 9, z = 0 \) in the \( xy \)-plane. The surface \( S \) is oriented so that the normal vector points outward. Compute the flux \( \iint_S \overrightarrow{F} \cdot d\overrightarrow{S} \) of the vector field
\[ \overrightarrow{F} = (z^2 + z + 1)\hat{i} + (-z + z^2 \sin (\sin(x)))\hat{j} + x^2y^2\hat{k}. \]

(A) \( 2\pi \)  
(B) \( 3\pi \)  
(C) \( 4\pi \)  
(D) \( 0 \)  
(E) \( 8\pi \)  
(F) \( \sqrt{3}\pi \)

**Answer 5.** If \( R \) is the 3D region enclosed by this closed surface, Gauss’s theorem gives
\[ \iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_R \nabla \cdot \overrightarrow{F} \, dV. \]
The divergence of \( \overrightarrow{F} \) is
\[ \nabla \cdot \overrightarrow{F} = \frac{\partial(z^2 + z + 1)}{\partial x} + \frac{\partial(-z + z^2 \sin (\sin(x)))}{\partial y} + \frac{\partial(x^2y^2)}{\partial z} = 0 + 0 + 0 = 0. \]
Substituting in the right hand side of Gauss’s theorem we get
\[ \iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_R \nabla \cdot \overrightarrow{F} \, dV = \iiint_R 0 \, dV = 0. \]
The correct answer is (D).

**Question 6.** Let \( C \) be the boundary of the part of the plane \( 2x + y + 2z = 2 \) in the first octant (oriented counterclockwise as viewed from above). Compute the line integral \( \int_C \overrightarrow{F} \cdot d\overrightarrow{r} \) of the vector field
\[ \overrightarrow{F} = (e^x + 3, \ x^2 + y - 1, \ z + 2). \]
(A) 2  (B) 0  (C) $e^2 - 1$

(D) $\frac{2}{3}$  (E) $3\pi$  (F) $\sqrt{8}$

**Answer 6.** The curve $C$ is the boundary of the triangle $S$ with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,1)$. The orientation of the triangle that is compatible with the counterclockwise orientation of $C$ corresponds to the normal vector to the plane, that is pointing inside the first octant. Using this orientation and Stokes’s theorem we can express the line integral of $\overrightarrow{F}$ as a flux through $S$:

$$
\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_S (\overrightarrow{\nabla} \times \overrightarrow{F}) \cdot d\overrightarrow{S}.
$$

To compute the right hand side we first compute the curl of $\overrightarrow{F}$:

$$
\overrightarrow{\nabla} \times \overrightarrow{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
e^x + 3 & x^2 + y - 1 & z + 2
\end{vmatrix} = \langle 0, 0, 2x \rangle.
$$

Next we have to parametrize $S$. From the equation of the plane we can get a parametrization by using say $x$ and $y$ as parameters and solving for $z$. We get that $S$ is parametrized by

$$
\overrightarrow{X}(x,y) = \langle x, y, 1 - x - \frac{1}{2}y \rangle
$$

with $0 \leq x \leq 1$, and $0 \leq y \leq 1 - \frac{x}{2}$. The normal vector for this parametrization is $\langle 1, 1/2, 1 \rangle$ and it is pointing inside the first octant so this parametrization gives $S$ the correct orientation.
Putting all of this together we can now compute
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \]
\[ = \iint_S (0, 0, 2x) \cdot d\mathbf{S} \]
\[ = \int_{x=0}^{1} \int_{y=0}^{2-2x} \langle 0, 0, 2x \rangle \cdot (1, \frac{1}{2}, 1) dydx \]
\[ = \int_{x=0}^{1} \int_{y=0}^{2-2x} 2x dydx \]
\[ = \int_{x=0}^{1} 2x (2-2x) dydx \]
\[ = \left( 2x^2 - \frac{4x^3}{3} \right) \Big|_{x=0}^{1} \]
\[ = \frac{2}{3}. \]

The correct choice is (D).

**Question 7.** True/False. Give a reason or a counterexample.

(i) The surface \( \overrightarrow{X}(u, v) = \langle u, v, 4-3u-2v \rangle \), \( 0 \leq u \leq 1, \ 0 \leq v \leq 1 \) is orientable.

(ii) The flux of the vector field \( \overrightarrow{F}(x, y, z) = x^2y\hat{i} + \sin(x + z)\hat{k} \) through a rectangular region \( R \) in the \( xz \) plane is equal to \( 2 \text{area}(R) \).

(iii) If \( S \) is a smooth orientable surface and \( \overrightarrow{F} \) is a smooth vector field, then value of the flux \( \iint_S \overrightarrow{F} \cdot d\overrightarrow{S} \) is independent of the choice of orientation on \( S \).

**Answer 7.** (i) is True. Indeed this is a smooth parametrized surface with no identification of points on the boundary of the parametrizing region in the \( uv \) plane. The normal vector for the parametrization thus gives an orientation of the surface.

(ii) is False. Indeed, the \( xz \) plane has equation \( y = 0 \) so \( R \) can be parametrized by \( x \) and \( z \) as parameters, and \( \overrightarrow{F} = \sin(x + z)\hat{k} \) on \( R \). The two possible orientations for \( R \) are then
given by the vectors \( \pm \hat{j} \) and so for the flux of \( \vec{F} \) through \( R \) we will have

\[
\iint_R \vec{F} \cdot d\vec{S} = \iint_R (\sin(x + z)\hat{k}) \cdot (\pm \hat{j}) dA = 0.
\]

In other words the flux is always zero and \textbf{not} equal to twice the area of \( R \) for either orientation.

(iii) is \textbf{False}. Indeed changing the orientation of \( S \) will change the sign of the normal vector and so the fluxes of \( \vec{F} \) for the two different orientations of \( S \) will differ by a minus sign. \( \square \)

Question 8. Find the surface area of the parametrized surface

\[
\vec{X}(s, t) = (3s \sin(t), 6s \sin(t) + 2s \cos(t), s \cos(t) + 1)
\]

with \( 0 \leq t \leq 2\pi \) and \( 0 \leq s \leq 1 \).

\begin{align*}
(A) & \ 6\pi & (B) & \ 2\pi & (C) & \ 3\pi \\
(D) & \ 2\pi - 3 & (E) & \ 8\pi & (F) & \ 9\pi
\end{align*}

Answer 8. To compute the area we will need the magnitude of the normal vector of the parametrization at each point of the surface. We compute

\[
\frac{\partial \vec{X}}{\partial s} = (3 \sin(t), 6 \sin(t) + 2 \cos(t), \cos(t)),
\]

\[
\frac{\partial \vec{X}}{\partial t} = (3 \cos(t), 6 \cos(t) - 2s \sin(t), -s \sin(t)),
\]

and so

\[
\vec{N} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 \sin(t) & 6 \sin(t) + 2 \cos(t) & \cos(t) \\ 3s \cos(t) & 6s \cos(t) - 2s \sin(t) & -s \sin(t) \end{bmatrix} = (-6s, 3s, -6s).
\]

Hence

\[
|\vec{N}(s, t)| = \sqrt{36s^2 + 9s^2 + 36s^2} = 9s,
\]

and so

\[
\text{area}(S) = \iint_S dS = \int_{s=0}^{1} \int_{t=0}^{2\pi} |\vec{N}(s, t)| \, dt \, ds = \int_{s=0}^{1} \int_{t=0}^{2\pi} 9s \, dt \, ds = 2\pi \int_{s=0}^{1} 9s \, ds = 2\pi \cdot \frac{9}{2} = 9\pi
\]

The correct answer is (F). \( \square \)