Math 241-002 Midterm 1
Fall 2013

Name:
Recitation Number and Day/Time:

Please *turn off and put away all electronic devices*. You may use both sides of a 8.5'' × 11'' sheet of paper for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. *Show all work*. Please *clearly mark* your final answer. Remember to put your name at the top of this page. Good luck!

My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this examination.

Your signature

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**Midterm 1**
**Total Score** /100
A Partial Table of Integrals

\[
\int_0^x u \cos nu \, du = \frac{\cos nx + nx \sin nx - 1}{n^2} \quad \text{for any real } n \neq 0
\]

\[
\int_0^x u \sin nu \, du = \frac{\sin nx - nx \cos nx}{n^2} \quad \text{for any real } n \neq 0
\]

\[
\int_0^x e^{mu} \cos nu \, du = \frac{e^{mx}(m \cos nx + n \sin nx) - m}{m^2 + n^2} \quad \text{for any real } n, m
\]

\[
\int_0^x e^{mu} \sin nu \, du = \frac{e^{mx}(-n \cos nx + m \sin nx) + n}{m^2 + n^2} \quad \text{for any real } n, m
\]

\[
\int_0^x \sin nu \cos mu \, du = \frac{m \sin nx \sin mx + n \cos nx \cos mx - n}{m^2 - n^2} \quad \text{for any real numbers } m \neq n
\]

\[
\int_0^x \cos nu \cos mu \, du = \frac{m \cos nx \sin mx - n \sin nx \cos mx}{m^2 - n^2} \quad \text{for any real numbers } m \neq n
\]

\[
\int_0^x \sin nu \sin mu \, du = \frac{n \cos nx \sin mx - m \sin nx \cos mx}{m^2 - n^2} \quad \text{for any real numbers } m \neq n
\]

Laplacian in polar coordinates

\[
\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.
\]
The temperature of a rod is described by the following equations:

\[
\begin{align*}
    u_t &= u_{xx} + e^{-x}, \quad 0 \leq x \leq 1, \quad t \geq 0 \\
    u(0,t) + 2u_x(0,t) &= 0 \\
    u_x(1,t) &= 3 \\
    u(x,0) &= \sin x
\end{align*}
\]

When it reaches equilibrium, what is the temperature at \( x = 0 \)?

(A) 0 \hspace{1cm} (B) \(-8 + 2e^{-1}\) \hspace{1cm} (C) \(-e^{-1}\)

(D) \(\sin(1) - e^{-1}\) \hspace{1cm} (E) \(\sin(1)\) \hspace{1cm} (F) \(3 + e^{-1}\)

**Answer:** The equilibrium temperature \( u(x) \) satisfies

\[
    u''(x) + e^{-x} = 0, \quad 0 \leq x \leq 1
\]

and the boundary conditions

\[
    u(0) + 2u'(0) = 0, \quad u(1) = 3.
\]

Integrating twice, we get

\[
    u(x) = -e^{-x} + c_1 x + c_2,
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants. The boundary conditions gives

\[
    -1 + c_2 + 2(1 + c_1) = 0
\]

and

\[
    e^{-1} + c_1 = 3,
\]

so \( c_1 = 3 - e^{-1}, \quad c_2 = -7 + 2e^{-1} \), and

\[
    u(x) = -e^{-x} + (3 - e^{-1})x + (-7 + 2e^{-1}).
\]

Plug in \( x = 0 \), we get

\[
    u(0) = -1 + (-7 + 2e^{-1}) = -8 + 2e^{-1}.
\]

The correct answer is (B).
Let \( u(x,t) \) be the solution of the equation \( u_t = 3u_{xx}, \ 0 \leq x \leq 3, \ t \geq 0 \) satisfying the boundary conditions

\[ u_x(0,t) = 0 \quad \text{and} \quad u_x(3,t) = 0 \]

\[ u(x,0) = 3 - \cos(3\pi x) \]

Compute \( u\left(\frac{1}{2},2\right)\).

(A) 0 \quad \text{(B) } \pi \quad \text{(C) } 3 - e^{-54\pi^2} \quad \text{(D) } 3 - e^{-162\pi^2} \quad \text{(E) } 3 \quad \text{(F) } -e^{-162\pi^2}

**Answer:** We use separation of variables, let

\( u(x,t) = \phi(x)G(t) \)

Substituting in the equation we get

\[ \frac{dG}{dt} \phi = 3G \frac{d^2\phi}{dx^2} \]

so we will have

\[ \frac{1}{3G} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\lambda \]

for some constant \( \lambda \).

The boundary conditions imply \( \phi'(0) = 0 \) and \( \phi'(3) = 0 \), so \( \phi \) must solve

\[ \phi'' + \lambda\phi = 0, \quad \phi'(0) = 0, \quad \phi'(3) = 0 \]

hence

\[ \lambda_n = \left( \frac{n\pi}{3} \right)^2, \ n = 0, 1, 2, \ldots \]

with the corresponding eigenfunction \( \phi_n \) given by

\[ \phi_n(x) = c \cdot \cos \frac{n\pi x}{3} \]

and for each eigenvalue \( \lambda_n \), \( G_n(t) \) satisfies

\[ G'_n = -3\lambda_n G_n \]

so

\[ G_n(t) = c \cdot e^{-3(\frac{n\pi}{3})^2 t} \]

By the principle of superposition, we form the general solution

\[ u(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{3} e^{-3(\frac{n\pi}{3})^2 t} \]
By initial condition \( u(x, 0) = 3 - 3 \cos(3\pi x) \), we can identify \( A_0 = 3 \), \( A_9 = -3 \), while other coefficients are 0. So
\[
u(x, t) = 3 - 3 \cos(3\pi x)e^{-27\pi^2 t} \]
Plug in \( x = \frac{1}{2}, t = 2 \), we get
\[
u\left(\frac{1}{2}, 2\right) = 3 - 3 \cos \frac{3\pi}{2}e^{-54\pi^2} = 3.
\]
So the correct answer is (E).
Let \( f(x) = x^2 - 4x \) for \( 0 \leq x \leq 2 \), and let 
\[
\sum_{n=0}^{\infty} a_n \cos \left( \frac{n \pi x}{2} \right)
\]
its Fourier cosine series. What is the value of \( a_4 \)?

\[
\begin{align*}
(A) \quad & \frac{1}{\pi^2} \\
(B) \quad & -\frac{1}{4\pi^3} \\
(C) \quad & 2 - \frac{1}{4\pi^3} \\
(D) \quad & -\frac{1}{\pi^2} - \frac{1}{4\pi^3} \\
(E) \quad & 3 \\
(F) \quad & \frac{1}{8\pi^3}
\end{align*}
\]

**Answer:** From the formula of Fourier cosine coefficients we compute

\[
a_4 = \frac{2}{\pi} \int_0^2 (x^2 - 4x) \cos \left( \frac{4\pi x}{2} \right) dx
\]

\[
= \frac{1}{2\pi} \int_0^2 (x^2 - 4x) d(\sin 2\pi x)
\]

\[
= \frac{1}{2\pi} (x^2 - 4x) \sin 2\pi x \bigg|_0^2 - \frac{1}{2\pi} \int_0^2 \sin 2\pi x d(x^2 - 4x)
\]

\[
= -\frac{1}{2\pi} \int_0^2 (2x - 4) \sin 2\pi x dx
\]

\[
= \frac{1}{4\pi^2} \int_0^2 (2x - 4) d(\cos 2\pi x)
\]

\[
= \frac{1}{4\pi^2} (2x - 4) \cos 2\pi x \bigg|_0^2 - \frac{1}{4\pi^2} \int_0^2 \cos 2\pi x d(2x - 4)
\]

\[
= \frac{1}{4\pi^2} (0 - (-4)) - \frac{1}{4\pi^2} \int_0^2 2 \cos 2\pi x dx
\]

\[
= \frac{1}{\pi^2} - \frac{1}{4\pi^2} \cdot \frac{2}{2\pi} \sin 2\pi x \bigg|_0^2
\]

\[
= \frac{1}{\pi^2} - \frac{1}{4\pi^3} (\sin 4\pi - \sin 0)
\]

\[
= \frac{1}{\pi^2}.
\]

The correct answer is (A).
Solve the Laplace equation of $u(r, \theta)$ on a 90° sector of a disk of radius 3:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 \leq r \leq 3, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$u(r, 0) = 0$$

$$u \left( r, \frac{\pi}{2} \right) = 0$$

$$|u(0, \theta)| < +\infty$$

$$u(3, \theta) = 1$$

Answer: Use separation of variables in polar coordinates, set

$$u(r, \theta) = G(r)\phi(\theta),$$

plug in the equation and divide by $\frac{1}{r^2} G\phi$, we get

$$-r \frac{d}{dr} \left( r \frac{dG}{dr} \right) - \frac{\lambda}{\phi} \frac{d^2 \phi}{d\theta^2} = -\lambda.$$  

The boundary conditions $u(r, 0) = 0$ and $u(r, \frac{\pi}{2}) = 0$ implies $\phi(0) = 0$ and $\phi(\frac{\pi}{2}) = 0$, so $\phi$ should solve the equation

$$\phi'' + \lambda \phi = 0$$

together with the boundary conditions

$$\phi(0) = 0, \quad \phi \left( \frac{\pi}{2} \right) = 0.$$  

We know the eigenvalues are

$$\lambda_n = 4n^2, \quad n = 1, 2, \cdots$$

and the corresponding eigenfunctions are

$$\phi_n(\theta) = c \cdot \sin 2n\theta.$$  

Plug each eigenvalue to the equation of $G$:

$$r \frac{d}{dr} \left( r \frac{dG}{dr} \right) = \lambda$$

we see that $G$ satisfies the Cauchy-Euler equation:

$$r(rG'')' = 4n^2G, \quad n = 1, 2, \cdots$$

so

$$G_n(r) = c_1 \cdot r^{2n} + c_2 \cdot r^{-2n}.$$  

Where $c_1$ and $c_2$ are two arbitrary constants. By $|u(0, \theta) < +\infty|$, solution has to be bounded at the origin, so $G$ cannot contain the $r^{-2n}$ term, therefore

$$G_n(r) = c \cdot r^{2n}.$$
By principle of superposition, the general solution is

\[ u(r, \theta) = \sum_{n=1}^{\infty} B_n r^{2n} \sin 2n\theta. \]

The condition \( u(3, \theta) = 1 \) gives

\[ \sum_{n=1}^{\infty} B_n 3^{2n} \sin 2n\theta = 1, \quad 0 \leq \theta \leq \frac{\pi}{2}, \]

we can compute \( B_n 3^{2n} \) by the formula of Fourier sine coefficients

\[ B_n 3^{2n} = \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} 1 \cdot \sin 2n\theta d\theta \]

\[ = \frac{4}{\pi} \left( -\frac{1}{2n} \cos 2n\theta \right) \bigg|_{0}^{\frac{\pi}{2}} \]

\[ = \frac{2}{n\pi} (1 - \cos n\pi) \]

\[ = \frac{2}{n\pi} (1 - (-1)^n) \]

So

\[ B_n = \frac{2}{3^{2n} n\pi} (1 - (-1)^n), \]

and

\[ u(r, \theta) = \sum_{n=1}^{\infty} \frac{2}{3^{2n} n\pi} (1 - (-1)^n) r^{2n} \sin 2n\theta. \]
(a) Compute the Fourier series of

\[ f(x) = \begin{cases} 
0, & \text{if } -\pi < x < 0, \\
x, & \text{if } 0 < x < \pi 
\end{cases} \]

on the interval \([-\pi, \pi]\). Fully simplify your answer - the formula for the coefficients should not contain any sines or cosines.

(b) What does this Fourier series converge to when \(x = \pi\)? Justify your answer.

---

**Answer:**

(a) Fourier series of \(f(x)\) on the interval \([-\pi, \pi]\) is

\[ a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \]

where

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \]
\[ = \frac{1}{2\pi} \int_{0}^{\pi} x \, dx \]
\[ = \frac{\pi}{4} \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \]
\[ = \frac{1}{\pi} \int_{0}^{\pi} x \cos nx \, dx \]
\[ = \frac{\cos n\pi + n\pi \sin n\pi - 1}{n^2} \]
\[ = \frac{(-1)^n - 1}{n^2\pi} \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \]
\[ = \frac{1}{\pi} \int_{0}^{\pi} x \sin nx \, dx \]
\[ = \frac{\sin n\pi - n\pi \cos n\pi}{n^2} \]
\[ = \frac{(-1)^{n+1}}{n} \]
So the Fourier series of $f(x)$ is

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \left( \frac{(-1)^n - 1}{n^2 \pi} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right).$$

(b) By convergence theorem, denote $\tilde{f}$ as the periodic extension of $f$, then the Fourier series converges at $x = \pi$ to

$$\frac{\tilde{f}(\pi^-) + \tilde{f}(\pi^+)}{2} = \frac{0 + \pi}{2} = \frac{\pi}{2}.$$