Math 241-002 Midterm 2
Fall 2013

NAME:

RECITATION NUMBER AND DAY/TIME:

Please turn off and put away all electronic devices. You may use both sides of a 8.5” × 11” sheet of paper for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. Show all work. Please clearly mark your final answer. Remember to put your name at the top of this page. Good luck!

My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this examination.

Your signature

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Midterm 2
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A Partial Table of Integrals

\[\int_0^x u \cos nu \, du = \frac{\cos nx + n x \sin nx - 1}{n^2} \quad \text{for any real } n \neq 0\]

\[\int_0^x u \sin nu \, du = \frac{\sin nx - n x \cos nx}{n^2} \quad \text{for any real } n \neq 0\]

\[\int_0^x e^{mu} \cos nu \, du = \frac{e^{mx}(m \cos nx + n \sin nx) - m}{m^2 + n^2} \quad \text{for any real } n, m\]

\[\int_0^x e^{mu} \sin nu \, du = \frac{e^{mx}(-n \cos nx + m \sin nx) + n}{m^2 + n^2} \quad \text{for any real } n, m\]

\[\int_0^x \sin nu \cos mu \, du = \frac{m \sin nx \sin mx + n \cos nx \cos mx - n}{m^2 - n^2} \quad \text{for any real numbers } m \neq n\]

\[\int_0^x \cos nu \cos mu \, du = \frac{m \cos nx \sin mx - n \sin nx \cos mx}{m^2 - n^2} \quad \text{for any real numbers } m \neq n\]

\[\int_0^x \sin nu \sin mu \, du = \frac{n \cos nx \sin mx - m \sin nx \cos mx}{m^2 - n^2} \quad \text{for any real numbers } m \neq n\]

Laplacian in polar coordinates

\[\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.\]
Let \( u(x,t) \) be a solution of the BVP

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \ t > 0, \\
u(0,t) &= 1, \\
u(\pi,t) &= 2\pi + 1, \\
u(x,0) &= 1 - \sin(x), \\
u_t(x,0) &= 0.
\end{align*}
\]

Find the value of \( u\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \).

(A) \( \frac{\pi}{2} \)  \quad (B) 0  \quad (C) \ -1  \\
(D) \( \frac{1}{\pi} \)  \quad (E) \( 1 + \pi \)  \quad (F) \( \frac{1}{4} - \frac{3\pi}{2} \)

**Answer 1.**

Since this equation has a non-homogeneous boundary condition, the solution will be equilibrium solution plus solution of a homogeneous problem.

Denote the equilibrium as \( v(x) \), it satisfies

\[
\begin{align*}
0 &= v''(x), \quad 0 < x < \pi \\
v(0) &= 1, \\
v(\pi) &= 2\pi + 1,
\end{align*}
\]

and we find \( v(x) = 2x + 1 \).

Now let \( w(x,t) = u(x,t) - v(x) \), then \( w \) satisfies the homogeneous equation

\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2} &= \frac{\partial^2 w}{\partial x^2}, \quad 0 < x < \pi, \ t > 0, \\
w(0,t) &= 0, \\
w(\pi,t) &= 0, \\
w(x,0) &= -\sin x - 2x, \\
w_t(x,0) &= 0.
\end{align*}
\]

and we get

\[
w(x,t) = \sum_{n=1}^{\infty} \sin(nx)(a_n \cos(nt) + b_n \sin(nt))
\]

where

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} (-\sin x - 2x) \sin(nx) \, dx
\]

and

\[
b_n = 0
\]
so

\[ u(x, t) = v(x) + w(x, t) = 2x + 1 + \sum_{n=1}^{\infty} a_n \sin(nx) \cos(nt) \]

and

\[ u\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \pi + 1 + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{2} \cos \frac{n\pi}{2} = \pi + 1 \]

The last identity holds because for any integer \( n \), either \( \sin \frac{n\pi}{2} \) or \( \cos \frac{n\pi}{2} \) is zero. The correct answer is (E).
Let $u(x, t)$ be the vertical displacement of a vibrating string of infinite length. The string has constant density $\rho = 1$ and tension with constant magnitude $T = 9$. The initial position $u(x, 0) = p(x)$ and velocity $u_t(x, 0) = v(x)$ are given by and

$$
p(x) = \begin{cases} 
1 - |2x - 1|, & \text{when } 0 < x < 1, \\
0, & \text{when } x < 0 \text{ or } x > 1,
\end{cases}
$$

and $v(x) = 0$ for all $x$. Calculate $u \left( \frac{1}{2}, \frac{1}{2} \right)$.

(A) $\frac{3}{2}$  (B) 0  (C) $\frac{3}{4}$  
(D) 1  (E) $\frac{1}{6}$  (F) $\frac{1}{2}$

Answer 2.

The equation of this vibrating string is

$$u_{tt} = 9u_{xx}$$

By d’Alembert’s formula,

$$u(x, t) = \frac{1}{2} (p(x + 3t) + p(x - 3t)) + \frac{1}{6} \int_{x-3t}^{x+3t} v(s) \, ds$$

notice $v(x) = 0$, so

$$u(x, t) = \frac{1}{2} (p(x + 3t) + p(x - 3t))$$

and

$$u \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2} (p(2) + p(-1)) = 0$$

So the correct answer is (B).
Consider the Sturm-Liouville equation

\[ \phi'' - 7\phi + \lambda(x^2 + 2)\phi = 0 \]

for a function \( \phi(x) \) defined for \( 0 \leq x \leq 1 \) with boundary conditions

\[ \phi(0) = 0, \quad \phi'(1) = 0. \]

Let \( \lambda_1 < \lambda_2 < \lambda_3 < \cdots \) be the set of all eigenvalues of the above equation, and let \( \phi_n(x) \) be an eigenfunction for the eigenvalue \( \lambda_n \) chosen so that \( \int_0^1 \phi_n^2(x)(x^2 + 2)dx = 1, \ n \geq 1 \). Which one of the following statements is true? Justify your reasoning.

(A) \( \int_0^1 \phi_n^2(x)dx = 0 \) for \( n \geq 1 \).

(B) \( \phi_4(x)\phi_5(x) > 0 \) for all \( 0 < x < 1 \).

(C) this is a singular Sturm-Liouville BVP.

(D) \( \lim_{n \to +\infty} \lambda_n = +\infty \).

(E) If \( n \gg 0 \), then \( |\phi_n(x)| > 0 \) for all \( 0 < x < 1 \).

(F) If \( a_n = \int_0^1 (2x - x^2)\phi_n(x)(x^2 + 2)dx \), then

\[ \sum_{n=1}^{\infty} a_n \phi_n \left( \frac{1}{2} \right) \]

converges to 2.

**Answer 3.**

(A) is not true because \( \phi_n(x) \) is continuous and not identically zero for \( n \geq 1 \), and therefore its square integration cannot be zero.

(B) is not true because \( \phi_n(x) \) has \( (n - 1) \) zeros in \( 0 < x < 1 \), and in particular, \( \phi_4(x)\phi_5(x) \) has at least 5 zeros in the interval \( (0, 1) \), thus cannot be always positive.

(C) is not true because we can identify

\[ p(x) = 1, \quad q(x) = -7, \quad \sigma(x) = x^2 + 2 \]

all of them are continuous in the closed interval \([0, 1]\), and \( p(x) > 0, \sigma(x) > 0 \), also the boundary conditions are homogeneous, hence this is a regular Sturm-Liouville BVP.

(D) is true because of Sturm-Liouville theorem.

(E) is not true because \( |\phi_n(x)| \) has \( (n - 1) \) zeros in \((0, 1)\) and cannot be always positive.

(F) is not true. Since \( \frac{1}{2} \) is a continuous point of the function \( 2x - x^2 \), according to Sturm-Liouville theorem, the series converges to the function plug in \( x = \frac{1}{2} \), which is \( \frac{3}{4} \).

So the correct answer is (D).
Explicitly show that the eigenvalue problem

$$\sqrt{1 + x^2} \phi'' + x \phi' = -\lambda \cdot 3 \sqrt{1 + x^2} \phi \quad \text{on} \quad [0, 1] \quad \text{with} \quad \phi(0) = \phi(1) = 0,$$

is a regular Sturm-Liouville problem. Write down the orthogonality condition on the eigenfunctions, and an asymptotic expression for the eigenvalues, valid as $\lambda \to \infty$.

Answer 4.

We want to multiply the equation by $f(x)$ so that it becomes a standard Sturm-Liouville equation, this requires $f(x)$ to satisfy

$$\left(\sqrt{1 + x^2} f(x)\right)' = x f(x)$$

which is

$$\sqrt{1 + x^2} f'(x) + \frac{x}{\sqrt{1 + x^2}} f(x) = x f(x)$$

so

$$\frac{f'(x)}{f(x)} = \frac{x}{\sqrt{1 + x^2}} - \frac{x}{1 + x^2}$$

notice the left hand side is $(\ln f(x))'$. Integrating about $x$, we get

$$\ln f(x) = \int \left(\frac{x}{\sqrt{1 + x^2}} - \frac{x}{1 + x^2}\right) dx = \sqrt{1 + x^2} - \frac{1}{2} \ln(1 + x^2)$$

and

$$f(x) = e^{\sqrt{1 + x^2} - \frac{1}{2} \ln(1 + x^2)} = \frac{e^{\sqrt{1 + x^2}}}{\sqrt{1 + x^2}}$$

Multiply the original differential equation by $f(x)$, we get

$$e^{\sqrt{1 + x^2}} \phi'' + \frac{x e^{\sqrt{1 + x^2}}}{\sqrt{1 + x^2}} \phi' + \lambda 3 e^{\sqrt{1 + x^2}} \phi = 0$$

therefore

$$p(x) = e^{\sqrt{1 + x^2}}, \quad q(x) = 0, \quad \sigma(x) = 3 e^{\sqrt{1 + x^2}}$$

we see that $p, q, \sigma$ are continuous in $[0, 1]$, and $p(x) > 0, \sigma(x) > 0$ as they are exponential functions, also the boundary conditions are homogeneous, so this is a regular Sturm-Liouville equation.

If we denote $\phi_n$ as the $n$-th eigenfunction, then the orthogonality condition is

$$\int_0^1 \phi_n \phi_m 3 e^{\sqrt{1 + x^2}} dx = 0, \quad m \neq n$$

Denote $\lambda_n$ as the $n$-th eigenvalue, then as $n \to \infty$

$$\lambda_n \sim \left(\frac{n \pi}{\int_0^1 \left(\frac{\sigma(x)}{p(x)}\right)^{\frac{1}{2}} dx}\right)^2 = \left(\frac{n \pi}{\int_0^1 \sqrt{3} dx}\right)^2 = \frac{n^2 \pi^2}{3}$$
Let 
\[ \sum_{n=-\infty}^{\infty} c_n e^{-inx} \]
be the complex form of the Fourier series of the function 
\[ f(x) = x + 1 \]
on the interval \([-\pi, \pi]\). What is the value of the sum \(c_{-2} + c_1\)?

**Answer 5.**
For \(n \neq 0\), we have
\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x + 1)e^{inx} \, dx \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} (x + 1) \frac{1}{in} e^{inx} \, dx \\
= \frac{1}{2n\pi i} (x + 1)e^{inx}\bigg|_{-\pi}^{\pi} - \frac{1}{2n\pi i} \int_{-\pi}^{\pi} e^{inx} \, dx \\
= \frac{1}{2n\pi i} (\pi + 1)e^{in\pi} - \frac{1}{2n\pi i} (-\pi + 1)e^{-in\pi} - \frac{1}{2n\pi i} \int_{-\pi}^{\pi} e^{inx} \, dx \\
= \frac{2\pi e^{in\pi}}{2n\pi i} - \frac{1}{2n\pi i} \int_{-\pi}^{\pi} e^{inx} \, dx \\
= \frac{e^{in\pi}}{ni} - \frac{1}{2n^2\pi^2 i^2} e^{inx}\bigg|_{-\pi}^{\pi} \\
= \frac{e^{in\pi}}{ni}
\]

So
\[
c_1 = \frac{e^{i\pi}}{i} = \frac{-1}{i} = i \\
c_{-2} = \frac{e^{-2\pi i}}{-2i} = \frac{1}{-2i} = \frac{i}{2}
\]
and
\[
c_{-2} + c_1 = \frac{i}{2} + i = \frac{3i}{2}
\]