11. The numbers 20604, 53227, 25755, 20927 and 289 are all divisible by 17. Show that the following determinant is also divisible by 17.

\[
\begin{vmatrix}
2 & 0 & 6 & 0 & 4 \\
5 & 3 & 2 & 2 & 7 \\
2 & 5 & 7 & 5 & 5 \\
2 & 0 & 9 & 2 & 7 \\
0 & 0 & 2 & 8 & 9 \\
\end{vmatrix}
\]

12. Let \( n \geq 2 \). Consider the operators

\[
d/dx, \quad d^2/dx^2 : \text{Pol}_n(\mathbb{R}) \to \text{Pol}_n(\mathbb{R}).
\]

True or False. Give a reason or a counter example.

(a) The operators \( d/dx \) and \( d^2/dx^2 \) have the same invariant subspaces in \( \text{Pol}_n(\mathbb{R}) \).

(b) The operators \( d/dx \) and \( d^2/dx^2 \) have the same eigenvectors in \( \text{Pol}_n(\mathbb{R}) \).

(c) The \( d/dx \) and \( d^2/dx^2 \) have the same eigenvalues.

13. Let \( A \in \text{Mat}_{n \times n}(\mathbb{C}) \) and \( B \in \text{Mat}_{m \times m}(\mathbb{C}) \) be fixed complex matrices. Consider the vector space \( V = \text{Mat}_{m \times n}(\mathbb{C}) \) and the linear operator

\[
T : V \to V, \quad T(X) = BXA.
\]

(a) Let \( b \in \mathbb{C}^m \) be an eigenvector of \( B \), and let \( a \in \mathbb{C}^n \) be an eigenvector of \( A^T \). Show that the \( m \times n \) matrix \( X = b \cdot a^T \) is an eigenvector for \( T \).

(b) Suppose \( A \) has distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \) and \( B \) has distinct eigenvalues \( \mu_1, \ldots, \mu_m \). Find all eigenvalues of \( T \) counting multiplicities.

(c) Suppose \( A \) has (not necessarily distinct) eigenvalues \( \lambda_1, \ldots, \lambda_n \) counting multiplicities, and suppose \( B \) has (not necessarily distinct) eigenvalues \( \mu_1, \ldots, \mu_m \) counting multiplicities. Find all eigenvalues of \( T \) counting multiplicities.

**Hint:** Show that you can find sequences of matrices \( \{A_k\}_{k=1}^\infty \subset \text{Mat}_{n \times n}(\mathbb{C}) \), \( \{B_\ell\}_{\ell=1}^\infty \subset \text{Mat}_{m \times m}(\mathbb{C}) \), so that all \( A_k \) and \( B_\ell \) have distinct eigenvalues and \( \lim_{k \to \infty} A_k = A, \lim_{\ell \to \infty} B_\ell = B \). Use this fact together with part (b).
14. Consider the **circulant** matrix

\[
A = \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\
a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
a_2 & a_3 & a_4 & \cdots & a_0 & a_1 \\
a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0
\end{pmatrix}
\]

associated with \( n \) numbers \( a_0, a_1, \ldots, a_{n-1} \).

(a) Let \( u_1, u_2, \ldots, u_n \) be the \( n \)-th roots of unity, that is the \( n \) distinct roots of the polynomial \( t^n - 1 \). Compute the product \( AW \), where \( W \) is the Wandermonde matrix

\[
W = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
u_1 & u_2 & u_3 & \cdots & u_{n-1} & u_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
u_1^{-2} & u_2^{-2} & u_3^{-2} & \cdots & u_{n-1}^{-2} & u_n^{-2} \\
u_1^{n-1} & u_2^{n-1} & u_3^{n-1} & \cdots & u_{n-1}^{n-1} & u_n^{n-1}
\end{pmatrix}
\]

(b) Use (a) and the multiplicativity of the determinant to show that \( \det(A) = f(u_1)f(u_2)\cdots f(u_n) \) where \( f(t) = a_0 + a_1t + a_1t^2 + \cdots + a_{n-1}t^{n-1} \).

(c) Find the eigenvalues and eigenvectors of \( A \).

15. True or false. Give a reason or a counter example.

(a) If \( A \in \text{Mat}_{n\times n}(\mathbb{C}) \), and \( \mathbf{v} \) is an eigenvector of \( A \) with eigenvalue \( \lambda \), then \( \mathbf{v} \) is an eigenvector of \( e^A \) with eigenvalue \( e^\lambda \).

(b) If \( F : V \rightarrow V \) is an operator on a finite dimensional complex vector space, then every \( F \)-invariant subspace contains an eigenvector for \( F \).

(c) Every permutation matrix in \( \text{Mat}_{n\times n}(\mathbb{C}) \) is diagonalizable.

(d) If \( P \in \text{Mat}_{n\times n}(\mathbb{C}) \) is a permutation matrix, then every eigenvalue of \( P \) is an eigenvalue of \( P^{-1} \).

(e) If a complex \( 5 \times 5 \) matrix \( A \) has two distinct eigenvalues, then \( A \) must have an eigenvalue with geometric multiplicity 2.
16. Consider the subspaces in $\mathbb{R}^4$:

$$U = \text{span} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \quad \text{and} \quad V = \text{span} \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}.$$  

Find bases of the subspaces $U + V$ and $U \cap V$ in $\mathbb{R}^4$.

17. Let $V$ be a finite dimensional real vector space and let $T : V \rightarrow V$ be a linear operator.

(a) Suppose that $T - \text{id}_V$ is nilpotent. Show that the operator $T$ is invertible.

(b) Suppose that there exists a polynomial $f(t) \in \text{Pol}(\mathbb{R})$ such that $f(0) \neq 0$ and such that $f(T) = 0$. Show that the operator $T$ is invertible.

18. Solve the initial value problem

\[
\begin{align*}
\frac{dx_1}{dt} &= 3x_1 + 2x_2 - 3x_3, \\
\frac{dx_2}{dt} &= 4x_1 + 10x_2 - 12x_3, \\
\frac{dx_3}{dt} &= 3x_1 + 6x_2 - 7x_3.
\end{align*}
\]

\[
\begin{align*}
x_1(0) &= 1, \\
x_2(0) &= -1, \\
x_3(0) &= 0.
\end{align*}
\]

19. (a) Find the Jordan canonical form of the matrix $J_n(0)^2$.

(b) Classify all nilpotent $5 \times 5$ complex matrices $A$ that have a square root.

(c) Classify all nilpotent $6 \times 6$ complex matrices that have a square root.

20. Let $V$ be an $n$-dimensional space over $\mathbb{C}$ and let $\Gamma \subset L(V, V)$ be a set of commuting operators. Show that there exists a vector $v \in V$ which is an eigenvector for all $T \in \Gamma$.  

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