Solutions to the Midterm Exam, Math 214, Spring 2020

Question 1. True or false. Give a reason or a counter-example

(a) If an $\mathbb{R}$-vector space has a finite generating set, then it is finite dimensional.

(b) A generating subset in a finite dimensional $\mathbb{R}$-vector space must consist of finitely many vectors.

(c) If $S$ is a finite set, and $\mathbb{K}$ is a field, then the vector space $\text{Fun}(S, \mathbb{K})$ of all functions from $S$ to $\mathbb{K}$ is finite dimensional.

Answer 1. Statement (a) is True because every finite generating set contains a maximal linearly independent subset and hence contains a basis.

Statement (b) is False since the set of all vectors in a vectors space is a spanning set. For instance if we view $V = \mathbb{R}$ as an $\mathbb{R}$-vector space, then $V$ contains infinitely many elements and they trivially generate $V$.

Statement (c) is True since the collection of delta functions $\{\delta_s\}_{s \in S}$ is a basis of $\text{Fun}(S, \mathbb{K})$. □

Question 2. Let $V$ be a vector space over a field $\mathbb{K}$, and let $x, y \in V$ be two vectors, and $a, b \in \mathbb{K}$ be two scalars. Show that

$$ax + by = bx + ay$$

if and only if $a = b$ and/or $x = y$. 
Answer 2. Since
\[ ax + by = bx + ay \]
the existence of additive inverses for vector addition gives
\[ ax + by - bx - ay = 0. \]
Commutativity of addition and distributivity of scaling and addition then give
\[ (a - b)(x - y) = 0. \]
If \( a - b \neq 0 \) we can multiply both sides of the last identity by \( 1/(a - b) \) which gives \( x - y = 0. \)

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Question 3. Which of the following subsets of vectors are vector subspaces. In each case either check the subspace properties or point out a property that fails and explain why.

(a) In the real 2-space \( \mathbb{R}^2 \) the subset \( S \subset \mathbb{R}^2 \) of all vectors with integral coordinates:
\[ S = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid a, b \in \mathbb{Z} \right\}. \]

(b) In the complex space \( \mathbb{C}^\infty \) of all sequences \((a_1, a_2, \ldots, a_n, \ldots)\) of complex numbers (with the term-by-term addition and scaling) the subset \( B \subset \mathbb{C}^\infty \) of all bounded sequences:
\[ B = \left\{ (a_i)_{i=1}^\infty \in \mathbb{C}^\infty \mid \text{there exists a positive real constant } c > 0 \text{ so that } |a_i| < c \text{ for all } i \right\}. \]

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Answer 3. In part (a) \( S \) is not a subspace. It is closed under addition but it is not closed under scaling. Specifically if we scale a vector with integral coordinates by a general real number we will get a vector with non-integral coordinates. For instance
\[ \sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}. \]
In part (b) \( S \) is a subspace. To check this suppose \( a = (a_i) \) and \( b = (b_i) \) are two bounded sequences of complex numbers and \( \alpha \) is a real number.

- Since \((a_i)\) is bounded we can find a positive real constant \( A \) so that \(|a_i| < A\) for all \( i = 1, 2, \ldots \). Similarly since \((b_i)\) is bounded we can find a positive real constant \( B \) so that \(|b_i| < B\) for all \( i = 1, 2, \ldots \).
Consider the sum \( a + b \). Since the sum of sequences is defined term by term it follows that
\[
a + b = (a_i + b_i)_{i=1}^{\infty}.
\]
But by the triangle inequality for the absolute value we have
\[
|a_i + b_i| \leq |a_i| + |b_i| < A + B,
\]
for all \( i = 1, 2, \ldots \). Therefore \( a + b \) is a bounded sequence as well. This shows that the sum in \( \mathbb{R}^\infty \) preserves the condition of being bounded.

• Since the scaling of a sequence is defined term by term we have that
\[
\alpha a = (\alpha \cdot a_i)_{i=1}^{\infty}.
\]
Then by the multiplicativity of the absolute value we have
\[
|\alpha \cdot a_i| = |\alpha| \cdot |a_i| < |\alpha| \cdot A
\]
for all \( i = 1, 2, \ldots \). This shows that scaling in \( \mathbb{R}^\infty \) preserves the condition of being bounded.

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**Question 4.** Let Pol be the vector space of all polynomials with real coefficients in one variable. Suppose that \( V \subset Pol \) is a vector subspace such that:

• For every \( k = 0, 1, 2, \ldots, n \) the subspace \( V \) contains a polynomial of degree exactly \( k \).

  In other words for every \( k = 0, 1, 2, \ldots, n \) we have a polynomial \( p_k(x) \in V \) such that
  \[ p_k(x) = c_k x^k + \text{lower degree terms}, \text{ and } c_k \neq 0. \]

• \( V \) does not contain any polynomials of degree \( > n \).

Show that \( V \) must be equal to the subspace \( Pol_n \subset Pol \) of polynomials of degree at most \( n \).

**Answer 4.** By assumption \( V \) does not contain any polynomials of degree \( > n \). Therefore \( V \subset Pol_n \). To show that \( V = Pol_n \) it suffices to check that \( V \) contains a set of polynomials that spans \( Pol_n \).

We are given polynomials \( p_0(x), p_1(x), \ldots, p_n(x) \) in \( V \) such that for every \( k = 0, 1, \ldots, n \) we have
\[
p_k(x) = c_k x^k + \text{lower degree terms}, \text{ and } c_k \neq 0.
\]
We can use these polynomials to argue that \( V \) contains all monomials \( 1, x, x^2, \ldots, x^n \).

We will argue by induction on \( n \).
**Base:** $n = 0$. We need to show that $1 \in V$. By assumption we know that we have a polynomial $p_0(x) \in V$ where 

$$p_0(x) = c_0, \quad \text{and} \quad c_0 \neq 0.$$ 

Since $V$ is a vector subspace we will have that $\frac{1}{c_0}p_0(x) \in V$ But $\frac{1}{c_0}p_0(x) = 1$ hence $1 \in V$.

**Step:** Suppose that we know that if $V$ contains polynomials $p_0(x), \ldots, p_{n-1}(x)$ satisfying 

$$p_k(x) = c_kx^k + \text{lower degree terms, with } c_k \neq 0.$$ 

for $k = 1, \ldots, n - 1$, then $V$ contains the monomials $1, x, \ldots, x^{n-1}$. Suppose in addition $V$ contains a polynomial $p_n(x)$ such that 

$$p_n(x) = c_nx^n + \text{lower degree terms, with } c_n \neq 0.$$ 

We need to show that $V$ contains the monomial $x^n$.

Explicitly 

$$p_n(x) = c_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \cdots a_1x + a_0,$$

and so 

$$x^n = \frac{1}{c_n}p_n(x) - \frac{a_{n-1}}{c_n}x^{n-1} - \cdots - \frac{a_1}{c_n}x - \frac{a_0}{c_n}.$$ 

Since $p_n(x) \in V$ and by the inductive assumption $1, x, \ldots, x^{n-1}$ it follows that the right hand side is a linear combination of polynomials in $V$. Since $V$ is a vector space this implies $x^n \in V$ and completes the check.

\[\Box\]

**Question 5.** Let $U \subset \text{Mat}_{2\times2}(\mathbb{R})$ be the subspace of all symmetric matrices and $V \subset \text{Mat}_{2\times2}(\mathbb{R})$ be the subspace of all strictly upper triangular matrices:

$$U = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \bigg| a, b, c \in \mathbb{R} \right\},$$

$$V = \left\{ \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \bigg| d \in \mathbb{R} \right\}.$$

(a) Show that $U \oplus V = \text{Mat}_{2\times2}(\mathbb{R})$.

(b) Decompose the matrix $E = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ into a sum $E = A + B$ with $A \in U$ and $B \in U$.

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**Answer 5.** For part (a) consider the subspace \( W := U + V \subset \text{Mat}_{2 \times 2}(\mathbb{R}) \). Note that \( U \cap V = \{0\} \). Indeed, if \( X \in U \cap V \) is a matrix which is both in \( U \) and \( V \), then on one hand we have
\[
X = \begin{pmatrix} a & b \\ b & c \end{pmatrix},
\]
and on the other
\[
X = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}.
\]
Therefore we must have \( b = d \), and \( a = 0 \), \( b = 0 \), and \( c = 0 \). This shows that \( W = U \oplus V \). But every matrix in \( U \) can be written uniquely as
\[
\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]
Therefore
\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\]
is a basis of \( U \) and so \( \dim U = 3 \). Similarly, note that every matrix in \( V \) is a scaling of \( \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \) and thus \( \dim V = 1 \). Since \( W = U \oplus V \) this implies that \( \dim W = \dim U + \dim V = 3 + 1 = 4 \). But \( \dim \text{Mat}_{2 \times 2}(\mathbb{R}) \) is also equal to 4 and since \( W \) is a subspace we must have \( W = \text{Mat}_{2 \times 2}(\mathbb{R}) \). This proves part (a).

For part (b) we need to solve the equation
\[
\begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}.
\]
This is equivalent to \( 1 = a \), \( 1 = b + d \), \( 2 = b \), and \( -1 = c \), and so we get
\[
\begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.
\]

\( \square \)

**Question 6.** Let \( S \) be a finite set and let \( V = (\mathcal{P}(S), +, \cdot) \) be the power set of \( S \) considered as a vector space over \( \mathbb{F}_2 \) where for \( A, B \subset S \), and \( \alpha \in \mathbb{F}_2 \) we have
\[
A + B = A \Delta B = A \cup B - A \cap B
\]
\[
\alpha \cdot A = \begin{cases} A, & \text{if } \alpha = 1, \\ \emptyset, & \text{if } \alpha = 0, \end{cases}
\]
Suppose that \( X, Y, Z \) are subsets in \( S \) such that \( X \not\subset Y \cup Z \), \( Y \not\subset X \cup Z \), and \( Z \not\subset X \cup Y \). Show that \( X, Y, \) and \( Z \) are linearly independent when viewed as vectors in \( V \).
This shows that \( X \not\subset Z \) and subsets in the (1) are never empty. \[ Y + X \not\subset Z. \]
Since in \( V \) the zero vector corresponds to the empty subset \( \emptyset \subset S \), we need to show that the subsets in the (1) are never empty.

First note that \( \emptyset \) is contained in every subset, and so the conditions \( X \not\subset Y \cup Z \), \( Y \not\subset X \cup Z \), and \( Z \not\subset X \cup Y \) imply that none of \( X \), \( Y \), and \( Z \) can be empty.

Let us examine \( X + Y \) next. By definition \( X + Y = (X \cup Y) - (X \cap Y) \) consists of all points in the union of \( X \) and \( Y \) which do not belong simultaneously in \( X \) and \( Y \). But we know that \( X \not\subset Y \cup Z \) so we know that there is a point \( x \in X \) which does not belong to \( Y \) and does not belong to \( Z \). Hence \( x \not\in X \cap Y \) and so \( x \in (X \cup Y) - (X \cap Y) \). This shows that \( (X \cup Y) - (X \cap Y) \) is not empty or equivalently that \( X + Y \not= 0 \). The same reasoning shows that \( X + Z \not= 0 \) and that \( Y + Z \not= 0 \).

Finally note that we chose \( x \in X \) such that \( x \not\in Y \) and \( x \not\in Z \). Thus \( x \in X + Y = (X \cup Y) - (X \cap Y \not\subset (X + Y) \cap Z \not\subset (X + Y) \cap Z \). Therefore \( x \in X + Y + Z = ((X + Y) \cup Z) - (X + Y) \cap Z \). This shows that \( X + Y + Z \not= \emptyset \) or equivalently \( X + Y + Z \not= 0 \).

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**Question 7.** Let \( V \) and \( W \) be real vector spaces with bases \( \mathcal{E} = \{ e_1, e_2, e_3 \} \) and \( \mathcal{F} = \{ f_1, f_2 \} \) respectively. Suppose that the linear map \( T : V \to W \) has matrix \( \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 6 \end{pmatrix} \). Find the matrix of \( T \) in the bases \( \mathcal{E}' = \{ e_1, e_1 + e_2, e_1 + e_2 + e_3 \} \) and \( \mathcal{F}' = \{ f_1, f_1 + f_2 \} \).

**Answer 7.** Write \( e'_1, e'_2, e'_3 \) for the elements of the basis \( \mathcal{E}' \) and \( f'_1, f'_2 \) for the elements of the basis \( \mathcal{F}' \). To compute the matrix of \( T \) in these bases we need to compute the coordinates of the vectors in the collection \( T(\mathcal{E}') \) in the basis \( \mathcal{F}' \). Using the matrix of \( T \) in the bases
and \( \mathbb{F} \) we compute
\[
T(e'_1) = T(e_1) = 0 \cdot f_1 + 3 \cdot f_2 \\
= 3f_2,
\]
\[
T(e'_2) = T(e_1 + e_2) = T(e_1) + T(e_2) = (3f_2) + (1 \cdot f_1 + 4 \cdot f_2) \\
= f_1 + 7f_2,
\]
\[
T(e'_3) = T(e_1 + e_2 + e_3) = T(e_1) + T(e_2) + T(e_3) = \\
= (3f_2) + (1 \cdot f_1 + 4 \cdot f_2) + (2 \cdot f_1 + 6 \cdot f_6) \\
= 3f_1 + 13f_2.
\]

This gives the vectors \( T(\mathbb{E}') \) in terms of the basis \( \mathbb{F} \). To get expressions for these vectors in terms of the basis \( \mathbb{F}' \) we need to solve for the vectors in \( \mathbb{F} \) in terms of the vectors in \( \mathbb{F}' \). This is straightforward:
since \( f'_1 = f_1 \) and \( f'_2 = f_1 + f_2 \) we get \( f_1 = f'_1 \) and \( f_2 = -f'_1 + f'_2 \). Substituting these expression in the previous formulas gives
\[
T(e'_1) = 3f_2 = -3f'_1 + 3f'_2, \\
T(e'_2) = f_1 + 7f_2 = -6f'_1 + 7f'_2, \\
T(e'_3) = 3f_1 + 13f_2 = -10f'_1 + 13f'_2.
\]

Hence the matrix of \( T \) in the bases \( \mathbb{E}' \) and \( \mathbb{F}' \) is
\[
\begin{pmatrix}
-3 & -6 & -10 \\
3 & 7 & 13
\end{pmatrix}.
\]

**Question 8.** Let \( V \) be a vector space over a field \( \mathbb{K} \) and let \( f : V \to \mathbb{K} \) be a linear function which is not identically zero. Consider the subspace \( U = \{ x \in V \mid f(x) = 0 \} \) and let \( a \in V \) be any vector that does not belong to \( U \).

(a) Show that for every vector \( v \in V \) the vector
\[
x = v - \frac{f(v)}{f(a)} a
\]

is well defined and belongs to \( U \).

(b) Show that \( U \oplus \text{span}(a) = V \).

**Answer 8.** For part (a) note that \( a \not\in U \) means \( f(a) \neq 0 \) in \( \mathbb{K} \). Therefore we can divide by \( f(a) \) in \( \mathbb{K} \) and so the vector
\[
x = v - \frac{f(v)}{f(a)} a
\]
is well defined. To check that this vector belongs to \( U \) we evaluate \( f \) on \( x \):
\[
f(x) = f \left( v - \frac{f(v)}{f(a)} a \right) = f(v) - \frac{f(v)}{f(a)} f(a) = f(v) - f(v) = 0.
\]
This shows that \( x \in U \).

For part (b) note that part (a) implies that any vector \( v \in V \) is equal to the sum
\[
v = x + \frac{f(v)}{f(a)} a,
\]
and that \( x \in U \). Since \( (f(v)/f(a)) \cdot a \) is a scaling of \( a \) it belongs to \( \text{span}(a) \) and so \( V = U + \text{span}(a) \).

To check that this is a direct sum we need to check that \( U \cap \text{span}(a) = \{0\} \).

Suppose \( x \in U \cap \text{span}(a) \). Then \( f(x) = 0 \) and \( x = \alpha a \) for some \( \alpha \in \mathbb{K} \). But then \( 0 = f(x) = f(\alpha a) = \alpha f(a) \). Since \( f(a) \neq 0 \) it follows that we must have \( \alpha = 0 \). This implies that \( x = 0 \cdot a = 0 \) and so \( U \cap \text{span}(a) = \{0\} \).

\( \square \)