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1 Problems

(1) Let $A$ be a $3 \times 3$ matrix whose entries are real numbers such that $A^2 = 0$. Show that $I_3 + A$ is invertible.

Solution: 2[1]

(2) Let $A$ and $B$ be symmetric $n \times n$ matrices. Prove that $AB$ is symmetric if and only if $AB = BA$.

Solution: 2[2]

(3) Let $a_1, a_2, \ldots, a_n$ be given real numbers. Calculate

$\det(A) = \det\begin{bmatrix}
    a_1 - a_2 & a_2 - a_3 & \ldots & a_{n-1} - a_n & a_n - a_1 \\
    a_2 - a_3 & a_3 - a_4 & \ldots & a_n - a_1 & a_1 - a_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{n-1} - a_n & a_n - a_1 & \ldots & a_{n-3} - a_{n-2} & a_{n-2} - a_{n-1} \\
    a_n - a_1 & a_1 - a_2 & \ldots & a_{n-1} - a_{n-2} & a_{n-1} - a_n
\end{bmatrix}$

Solution: 2[3]

(4) Let $G$ be an abelian group and $n$ a fixed positive integer. Show that the subset of all elements in $G$ whose order divides $n$ is a subgroup in $G$. Will this be true if $G$ is non-abelian?

Solution: 2[4]
(5) Determine the automorphism group of a cyclic group of order 10.

**Solution:** 25

(6) Let $G$ be a group and let $x \in G$ be a fixed element. Consider the set $Z = \{ y \in G | xy = yx \}$.

(a) Show that $Z$ is a subgroup of $G$.

(b) Let $H < G$ be the cyclic subgroup generated by $x$. Show that $H$ is a normal subgroup of $Z$.

**Solution:** 26

(7) Consider the subsets $H$ and $N$ of $SL_2(\mathbb{R})$ defined by

$$N = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) \mid ac = 0 \text{ and } bd = 0 \right\}$$

and

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid a \neq 0 \right\}$$

Show that $N$ is a subgroup of $SL_2(\mathbb{R})$ and $H$ is a subgroup of $H$.

**Solution:** 27

(8) Let $G$ be an abelian group of order 12, and let $\varphi : G \to G$ be the homomorphism given by $\varphi(x) = x^{11}$ for all $x \in G$. Show that $\varphi$ is an isomorphism.

**Solution:** 28
(9) The numbers 20604, 53227, 25755, 20927, and 289 are all divisible by 17. Use Cramer’s rules to show that

\[
\begin{vmatrix}
2 & 0 & 6 & 0 & 4 \\
5 & 3 & 2 & 2 & 7 \\
2 & 5 & 7 & 5 & 5 \\
2 & 0 & 9 & 2 & 7 \\
0 & 0 & 2 & 8 & 9
\end{vmatrix}
\]

is also divisible by 17.

Solution: 2.9

(10) Consider the elements

\[x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \quad y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}\]

in the symmetric group \(S_4\) of permutations on four letters. Show that \(x\) and \(y\) are not conjugate in \(S_4\), i.e. show that there is no element \(\sigma \in S_4\) satisfying \(y = \sigma x \sigma^{-1}\). (Hint: Compute the sign of \(x\) and \(y\).)

Solution: 2.10

(11) Let \(G\) be any group. Let \(a \in G\) be an element of order 15. Show that there exists an element \(x \in G\) such that \(x^7 = a\).

Solution: 2.11

(12) Let \(A \in \text{Mat}_{m \times n}(\mathbb{R})\), and let \(B \in \text{Mat}_{n \times m}(\mathbb{R})\). Suppose \(AB = I_m\) and \(BA = I_n\). Show that \(m = n\).

Solution: 2.12
2 Solutions

Solution of problem 1: Recall the standard algebraic formula $x^2 - y^2 = (x + y)(x - y)$.

Now thinking of $a$ and $b$ as matrices and substituting $I_3$ for $x$ and $A$ for $y$ we get

$$ (I_3 + A)(I_3 - A) = I_3^2 - A^2 = I_3. $$

In other words $I_3 - A$ is the inverse matrix of $I_3 + A$.

Solution of problem 2: $AB$ is symmetric if and only if $AB = (AB)^t = B^t A^t$. Since both $A$ and $B$ are symmetric we have $B^t = B$ and $A^t = A$ and so $AB$ is symmetric if and only if $AB = BA$.

Solution of problem 3: Adding to any row of a matrix a multiple of another row does not change the determinant. So the determinant of $A$ is equal to the determinant of the matrix obtained from $A$ by replacing the last row by the sum of all rows. But the sum of all rows has all entries equal to zero and so $\det(A) = 0$.

Solution of problem 4: Let

$$ H = \{x \in G \mid x \text{ is of order dividing } n\}, $$

and let $a$ and $b$ be two elements in $H$. We need to check that $ab \in H$, and that $a^{-1} \in H$.

To check that $ab \in H$ we need to compute the order of $ab$. First note that $a, b \in H$ implies $a^n = b^n = e$. Moreover, since $G$ is abelian we
have
\[(ab)^n = (ab) \cdot (ab) \cdots (ab)\]
\[\text{\text{n times}}\]
\[= a^n \cdot b^n\]
\[= e \cdot e\]
\[= e.\]

Therefore \(ab\) has order dividing \(n\).

Similarly
\[(a^{-1})^n \cdot a^n = (a \cdot a^{-1})^n = e^n = e.\]

But \(a^n = e\) so we get that \((a^{-1})^n \cdot e = e\), i.e. \((a^{-1})^n = e\). Again this means that the order of \(a^{-1}\) divides \(n\).

This reasoning will not work if we can not switch the order of multiplication of \(a\) and \(b\). This does not prove that the statement can not be true in a non-abelian group since there could be some other reasoning that yields the statement. So, to show that the statement does not hold for non-abelian groups we have to exhibit a counter example.

Consider the simplest non-abelian group \(S_3\) and let \(H \subset S_3\) be the subset of all elements of order dividing 2. Then
\[S_3 = \{1, (12), (13), (23), (123), (132)\},\]
\[H = \{1, (12), (13), (23)\}.\]

But \((13)(12) = (123)\) which is of order 3. Hence \(H\) is not a subgroup.

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**Solution of problem 5:** Let \(C_{10} = \{1, x, x^2, \ldots, x^9\}\) be a cyclic group of order ten. If \(\phi : C_{10} \rightarrow C_{10}\) is an automorphism, then \(\phi\) is completely determined by the element \(\phi(x)\). The surjectivity of \(\phi\) implies that all elements in \(C_{10}\) should be powers of \(\phi(x)\) and so \(\phi(x)\) must generate \(C_{10}\). In particular this means that \(\phi(x)\) has order 10. On the other hand \(\phi(x) \in C_{10}\) so we can write \(\phi(x) = x^k\) with \(0 \leq k \leq 9\). But we
know that \( x^k \) has order 10 if and only if \( k \) and 10 are relatively prime. So we conclude that 1, 3, 7, 9 are the only possible values of \( k \).

Since \( \phi \) is uniquely determined by \( \phi(x) \) we see that \( \text{Aut}(C_{10}) \) contains exactly four elements \( \phi_1, \phi_3, \phi_7, \) and \( \phi_9 \), where

\[
\phi_1 : C_{10} \to C_{10}, \quad \phi_1(x^a) = x^a \text{ for all } a,
\]

\[
\phi_3 : C_{10} \to C_{10}, \quad \phi_3(x^a) = x^{3a} \text{ for all } a,
\]

\[
\phi_7 : C_{10} \to C_{10}, \quad \phi_7(x^a) = x^{7a} \text{ for all } a,
\]

\[
\phi_9 : C_{10} \to C_{10}, \quad \phi_9(x^a) = x^{9a} \text{ for all } a.
\]

Since \( \phi_1 \) is the identity map, it is the unit element of \( \text{Aut}(C_{10}) \). Furthermore \( \phi_3 \circ \phi_3(x) = \phi_3(\phi_3(x)) = \phi_3(x^3) = x^9 \), i.e. \( \phi_3 \circ \phi_3 = \phi_9 \). Similarly \( \phi_3 \circ \phi_3 \circ \phi_3(x) = \phi_3(\phi_3 \circ \phi_3(x)) = \phi_3(x^9) = x^{27} = x^7 \), and so \( \phi_3 \circ \phi_3 \circ \phi_3 = \phi_7 \). Finally \( \phi_3 \circ \phi_3 \circ \phi_3 \circ \phi_3(x) = \phi_3(\phi_3 \circ \phi_3 \circ \phi_3(x)) = \phi_3(x^7) = x^{21} = x \), i.e. \( \phi_3 \circ \phi_3 \circ \phi_3 \circ \phi_3 = \phi_1 \).

This shows that

\[ \text{Aut}(C_{10}) = \{ \phi_1, \phi_3, (\phi_3)^2, (\phi_3)^3 \}, \]

i.e. \( \text{Aut}(C_{10}) \) is a cyclic group of order 4.

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**Solution of problem 6:** If \( a, b \in Z \), then \( (ab)x = abx = axb = x(ab) \) and so \( ab \in Z \). Clearly \( 1 \cdot x = x = x \cdot 1 \) and so \( 1 \in Z \). Finally if \( a \in Z \), then \( a^{-1}xa = a^{-1}ax = x \). Multiplying this identity by \( a^{-1} \) on the right we get \( a^{-1}x = xa^{-1} \) which yields \( a^{-1} \in Z \).

Let now \( a \in Z \) and let \( k \in Z \). If \( k > 0 \), then \( ax^k = ax \cdot x^{k-1} = x \cdot ax^{k-1} = \ldots = x^ka \). Multiplying the last identity by \( x^{-k} \) on the left and on the right gives also \( ax^{-k} = x^{-k}a \) and so the cyclic subgroup generated by \( x \) is normal in \( Z \).
Solution of problem 7: By definition we have

\[
N = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) \mid ac = 0 \text{ and } bd = 0 \right\}
\]

\[
= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) \mid \text{either } a = d = 0, c = -b^{-1} \text{ or } b = 0, c = 0, d = a^{-1} \right\}
\]

In particular \( H \subset N \). Now we compute

\[
\begin{bmatrix} 0 & b_1 \\ -b_1^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & b_2 \\ -b_2^{-1} & 0 \end{bmatrix} = \begin{bmatrix} -b_1/b_2 & 0 \\ 0 & -b_2/b_1 \end{bmatrix}
\]

\[
\begin{bmatrix} 0 & b \\ -b^{-1} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -b \\ b^{-1} & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} a_1 & 0 \\ a_1^{-1} \end{bmatrix} \cdot \begin{bmatrix} a_2 & 0 \\ a_2^{-1} \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 \\ a_1^{-1}a_2^{-1} \end{bmatrix}
\]

\[
\begin{bmatrix} a & 0 \\ a^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & 0 \\ a \end{bmatrix}
\]

This shows that \( H \subset N \subset SL_2(\mathbb{R}) \).

Solution of problem 8: Since \( G \) is finite its suffices to check that \( \varphi \) is injective. Write \( e \) for the identity element in \( G \). Let \( x^a \) and \( x^b \) be two elements in \( G \) such that \( \varphi(x^a) = \varphi(x^b) \). Then \( x^{11a} = x^{11b} \) and so the order of \( x^{a-b} \) must divide 11. On the other hand \( x^{a-b} \) is an element in \( G \) and so its order divides \( |G| = 12 \). Since 11 and 12 are coprime, it follows that the only positive integer that divides the both 11 and 12 is 1, i.e. the order of \( x^{a-b} \) is 1 or equivalently \( x^{a-b} = e \). This shows that \( x^a = x^b \) and so \( \varphi \) is injective.
Solution of problem 9: If \( \det A = 0 \) then 17 divides \( \det A \) and there is nothing to prove. Assume \( \det A \neq 0 \). We have

\[
\begin{bmatrix}
2 & 0 & 6 & 0 & 4 \\
5 & 3 & 2 & 2 & 7 \\
2 & 5 & 7 & 5 & 5 \\
2 & 0 & 9 & 2 & 7 \\
0 & 0 & 2 & 8 & 9
\end{bmatrix}
\begin{bmatrix}
10^4 \\
10^3 \\
10^2 \\
10 \\
1
\end{bmatrix}
= \begin{bmatrix}
20604 \\
53227 \\
25755 \\
20927 \\
289
\end{bmatrix}.
\]

In other words, the vector

\[
X = \begin{bmatrix}
10^4 \\
10^3 \\
10^2 \\
10 \\
1
\end{bmatrix}
\]

is a solution of the linear system

\[AX = B,\]

where

\[
A = \begin{bmatrix}
2 & 0 & 6 & 0 & 4 \\
5 & 3 & 2 & 2 & 7 \\
2 & 5 & 7 & 5 & 5 \\
2 & 0 & 9 & 2 & 7 \\
0 & 0 & 2 & 8 & 9
\end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix}
20604 \\
53227 \\
25755 \\
20927 \\
289
\end{bmatrix}.
\]

By Cramer’s formulas we have

\[
\begin{bmatrix}
10^4 \\
10^3 \\
10^2 \\
10 \\
1
\end{bmatrix}
= \begin{bmatrix}
\det A_1 / \det A \\
\det A_2 / \det A \\
\det A_3 / \det A \\
\det A_4 / \det A \\
\det A_5 / \det A
\end{bmatrix},
\]

where \( A_i \) is the matrix obtained from \( A \) by replacing the \( i \)-th column by \( B \). But \( B = 17B' \), where \( B' \) is a vector with integer entries. Thus \( \det A_5 = 17 \det A'_5 \) where \( A'_5 \) is the matrix obtained from \( A \) by replacing the fifth column by \( B' \). Since all entries of \( B' \) are integers, it follows
that all entries of \( A'_5 \) are integers and so \( \det A'_5 \) is an integer. From Cramer’s formulas we have

\[
17 \det A'_5 = \det A,
\]

and so 17 divides \( \det A \).

---

**Solution of problem 10:** If we can find a \( \sigma \) in \( S_4 \) satisfying \( y = \sigma x \sigma^{-1} \), then \( \text{sgn}(y) = \text{sgn}(\sigma x \sigma^{-1}) = \text{sgn}(\sigma) \text{sgn}(x) \text{sgn}(\sigma)^{-1} \) since

\[
\text{sgn} : S_4 \to \{\pm 1\}
\]

is a homomorphism. Furthermore, \( \{\pm 1\} \) is abelian and so

\[
\text{sgn}(\sigma) \text{sgn}(x) \text{sgn}(\sigma)^{-1} = \text{sgn}(\sigma) \text{sgn}(\sigma)^{-1} \text{sgn}(x) = \text{sgn}(x).
\]

This shows that if \( x \) and \( y \) are conjugate, then we must have \( \text{sgn}(x) = \text{sgn}(y) \). On the other hand, by definition we have \( \text{sgn}(x) = \det(P_x) \) and \( \text{sgn}(y) = \det(P_y) \), where \( P_x \) and \( P_y \) are the permutation matrices

\[
P_x = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\]

and so \( \det(P_x) = -1 \) and \( \det(P_y) = 1 \), i.e. \( x \) and \( y \) have different signs. This contradicts the assumption that \( x \) and \( y \) are conjugate.

---

**Solution of problem 11:** The greatest common divisor of 7 and 15 is 1. Therefore we can find integers \( u \) and \( v \) such that \( 7u + 15v = 1 \). This follows from the classification of subgroups in \( \mathbb{Z} \) but in this case we can find \( u \) and \( v \) explicitly: \( 7 \cdot (-2) + 15 \cdot 1 = 1 \). Now raising \( a \) in this power we compute

\[
a = a^1 = a^{7 \cdot (-2) + 15 \cdot 1} = (a^{-2})^7 \cdot a^{15} = (a^{-2})^7 \cdot e = (a^{-2})^7.
\]
Therefore we can take $x = a^{-2}$.

Solution of problem 12: The rank of $A$ is equal to the number of non-zero rows in the row-reduced echelon form $R$ of $A$. Since $R$ is obtained from $A$ by left multiplication by elementary matrices, it follows that $RB$ is obtained from $AB$ by left multiplication by elementary matrices. Thus $AB$ and $RB$ are row equivalent. If $R$ has any zero rows at the bottom, then $RB$ will have zero rows at the bottom. Thus rank$(AB) = \text{rank}(RB) \leq \text{rank}(R) = \text{rank}(A)$. Therefore rank$(A) \geq m$. Similarly rank$(BA) \leq \text{rank}(B)$ and hence rank$(B) \geq n$. On the other hand rank$(A) \leq \min m, n$ since the rank of $A$ is equal to the number of pivots in the row reduced echelon form of $A$ and by the same reasoning rank$(B) \leq \min m, n$. This implies rank$(A) = \text{rank}(B) = m = n$. G