Math 371, practice problems for exam 2

1. Let R be a commutative unital ring. Show that if $P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots \supseteq P_k \supseteq \cdots$ is decreasing sequence of prime ideals, then $\cap_{i=1}^{\infty} P_i$ is also a prime ideal.

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2. The following steps show that there exists an integral domain that contains a pair of elements with no gcd. Let A be the subset in $\mathbb{R}[x]$ that consists of all polynomials with no linear term, that is $f(x) \in A$ if $f(x) = a_0 + a_2x^2 + a_3x^3 + \cdots$.

(a) Prove that A is a subring in $\mathbb{R}[x]$ and that A is an integral domain.

(b) Prove that x^5 and x^6 have no gcd in A.

(c) Is x^2 irreducible in A? Is it prime?

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3. Let $I \triangleleft \mathbb{Z}[x]$ be a maximal ideal. Prove that the field $\mathbb{Z}[x]/I$ is finite.

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4. True or false. Give a reason or a counter example.

(a) There exists a field K which strictly contains the field of complex numbers as a subfield.

(b) Every finite field has positive characteristic.

(c) There exists an infinite field of characteristic 3.

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5. Let K be a field. Prove that the ring $F[x, x^{-1}]$ of Laurent polynomials is a principal ideal domain.

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6. Let A be the subring in the ring of all real valued functions on the interval $[-\pi, \pi]$ consisting of functions that can be represented as polynomials of $\cos x$ and $\sin x$.

(a) Show that A is an integral domain.

(b) Show that $A \cong \mathbb{R}[u, v]/(u^2 + v^2 - 1)$.

(c) Show that $\mathrm{Frac}(A)$ is isomorphic to the field $\mathbb{R}(t)$ of rational functions over $\mathbb{R}$.

7. Determine $\mathrm{Frac}(R)$ if R is the subring of $\mathbb{Z}[x]$ consisting of all polynomials f such that $f'(0) = f''(0) = 0$.

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8. Which of the rings $\mathbb{Z}[1/28]$, $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}[x]$ are UFDs? Inthose that are factor 2, 3, 4, 5, and 6.

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9. Describe the fields of fractions of the rings $\mathbb{Z}[i]$, $\mathbb{Z}/(7)$, $\mathbb{Z}\left[e^{2\pi i/3}\right]$, $\mathbb{Q}[[x]]$.

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10. Find all $a \in \mathbb{F}_7$ for which the ring $\mathbb{F}_7[x]/(x^3 + a)$ is a field. Find all $a \in \mathbb{F}_7$ for which the ring $\mathbb{F}_7[x]/(x^2 + a)$ is a field.
