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1 Problems

(1) Let $V$ be a finite dimensional unitary space and $f : V \to V$ be a self-adjoint operator. Fix a complex number $c \in \mathbb{C}$ with $\text{Im}(c) > 0$.

(a) Show that the operator $f - c \cdot \text{id}$ is invertible.
(b) Show that the operator

$$g := (f - \bar{c} \cdot \text{id}) \cdot (f - c \cdot \text{id})^{-1}$$

unitary.
(c) Show that $g - \text{id}$ is invertible.
(d) Check that

$$f = (c \cdot g - \bar{c} \cdot \text{id})(g - \text{id})^{-1}.$$ 

Solution: 2.1

(2)

(a) Show that the usual exponential map

$$\exp : \mathbb{R} \to \mathbb{R}_{>0}$$

is an isomorphism of groups.
(b) Show that the complex exponential

$$\exp : \mathbb{C} \to \mathbb{C}^\times$$

is a surjective group homomorphism. Is it an isomorphism?
(c) Check if the matrix exponential

$$\exp : \mathfrak{gl}(n, \mathbb{C}) \to GL(n, \mathbb{C})$$

is a group homomorphism.
Solution: 2.2

(3) Let $\mathbb{E}$ denote the Euclidean three space with standard coordinates $(x, y, z)$.

Recall that physically $\mathbb{E}$ is thought of as the space of all self-adjoint traceless operators on the space $\mathbb{H}$ of spinor states. A direction in $\mathbb{E}$ is a ray emanating from the origin. Every such ray can be written as $\mathbb{R}_{>0}f$ where $f \in \mathbb{E} \subset \mathcal{L}(\mathbb{H}, \mathbb{H})$. As we showed in class, specifying a direction in $\mathbb{E}$ is equivalent to specifying a decomposition $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$ of $\mathbb{H}$ of two orthogonal one dimensional subspaces, where $\mathbb{H}_+$ is the eigenspace of $f$ corresponding to the positive eigenvalue of $f$ and $\mathbb{H}_-$ is the eigenspace corresponding to the negative eigenvalue of $f$. The spinor state $\mathbb{H}_+$ is called the state of spin projection $1/2$ in the direction $\mathbb{R}_{>0}f$.

Consider an electron in $\mathbb{E}$ whose spin projection on the positive direction of the $y$-axis is $1/2$. What is the probability that this electron will pass through a Stern-Gerlah machine filtering only electrons with spin projection $1/2$ on the positive direction of the $z$-axis?

Solution: 2.3

(4) Let $V$ be a unitary space and let $f : V \to V$ be any linear operator.

(a) Show that $f^*f$ and $ff^*$ are self-adjoint operators.

(b) Show that $f^*f$ and $ff^*$ are positive definite if and only if $f$ is invertible.
(5) Let $Q$ be the algebra of quaternions. By definition $Q$ is a real four dimensional vector space with basis $1, i, j$ and $k$ and distributive associative product given by the multiplication table:

\[
\begin{align*}
1^2 &= 1, & 1i &= i, & 1j &= j, & 1k &= k, \\
i1 &= i, & i^2 &= -1, & ij &= k, & ik &= -j \\
j1 &= j, & ji &= -k, & j^2 &= -1, & jk &= i \\
k1 &= k, & ki &= j, & kj &= -i, & k^2 &= -1
\end{align*}
\]

For every quaternion $\alpha = a1 + bi + cj + dk \in Q$ define the conjugate quaternion $\bar{\alpha}$ by the formula

\[\bar{\alpha} = a1 - bi - cj - dk.\]

(a) Compute $\alpha \bar{\alpha}$.

(b) Prove that every $\alpha \neq 0$ has a multiplicative inverse.

Solution: 2.5
2 Solutions

Solution of problem 1.1: (a) The operator $f - c \cdot \text{id}$ can fail to be invertible only if $c$ is an eigenvalue of $f$. However, since $f$ is self-adjoint, it only has real eigenvalues and so $c$ can not be an eigenvalue of $f$, in view of $\text{Im}(c) > 0$.

(b) Since $f$ is self-adjoint, it has only real eigenvalues and we can find an orthonormal basis $e_1, e_2, \ldots, e_n$ of $V$ consisting of eigenvectors of $f$. Thus $f(e_i) = \lambda_i e_i$ with $\lambda_1, \ldots, \lambda_n$ real. In particular we get

$$
(f - \bar{c} \cdot \text{id})(e_i) = (\lambda_i - \bar{c}) \cdot e_i
$$
$$
(f - c \cdot \text{id})^{-1}(e_i) = (\lambda_i - c)^{-1} \cdot e_i.
$$

In other words, the operator $(f - \bar{c} \cdot \text{id})(f - c \cdot \text{id})^{-1}$ diagonalizes in the orthonormal basis $\{e_i\}$ and has eigenvalues

$$
\frac{\lambda_1 - \bar{c}}{\lambda_1 - c}, \frac{\lambda_2 - \bar{c}}{\lambda_2 - c}, \ldots, \frac{\lambda_n - \bar{c}}{\lambda_n - c}.
$$

Therefore in order to check that $(f - \bar{c} \cdot \text{id})(f - c \cdot \text{id})^{-1}$ is unitary, we only need to show that all the complex numbers

$$
\frac{\lambda_i - \bar{c}}{\lambda_i - c}
$$

have modulus 1. Using the fact that $\lambda_i$ is real we compute

$$
\left| \frac{\lambda_i - \bar{c}}{\lambda_i - c} \right|^2 = \left( \frac{\lambda_i - \bar{c}}{\lambda_i - c} \right) \left( \frac{\lambda_i - \bar{c}}{\lambda_i - c} \right) = \left( \frac{\lambda_i - \bar{c}}{\lambda_i - c} \right) \left( \frac{\bar{\lambda}_i - c}{\lambda_i - \bar{c}} \right) =
$$
$$
= \left( \frac{\lambda_i - \bar{c}}{\lambda_i - c} \right) \left( \frac{\lambda_i - c}{\lambda_i - \bar{c}} \right) = 1,
$$

and so $g = (f - \bar{c} \cdot \text{id})(f - c \cdot \text{id})^{-1}$ must be unitary.
(c) Again \( g - \text{id} \) can fail to be invertible only if 1 is an eigenvalue of \( g \). This means that for some \( i \) we ought to have

\[
\frac{\lambda_i - \bar{c}}{\lambda_i - c} = 1
\]

or equivalently \( \bar{c} = c \). Since \( \text{Im}(c) > 0 \) this is impossible.

(d) Applying the operator \((c \cdot g - \bar{c} \cdot \text{id})(g - \text{id})^{-1}\) to the vector \( e_i \) we see that \( e_i \) is an eigenvector of \((c \cdot g - \bar{c} \cdot \text{id})(g - \text{id})^{-1}\) with eigenvalue

\[
\left( c \cdot \left( \frac{\lambda_i - \bar{c}}{\lambda_i - c} \right) - \bar{c} \right) \cdot \left( \frac{\lambda_i - \bar{c}}{\lambda_i - c} - 1 \right)^{-1}
\]

\[
= \frac{c\lambda_i - \bar{c}\lambda_i - c\bar{c}}{\lambda_i - c} \cdot \left( \frac{\lambda_i - \bar{c} - \lambda_i + c}{\lambda_i - \bar{c}} \right)^{-1}
\]

\[
= \lambda_i \cdot \frac{c - \bar{c}}{\lambda_i - c} \cdot \frac{\lambda_i - c}{c - \bar{c}} = \lambda_i.
\]

Thus \( f \) and \((c \cdot g - \bar{c} \cdot \text{id})(g - \text{id})^{-1}\) have the same eigenvectors and eigenvalues and so must be equal.

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**Solution of problem 1.2:** (a) To show that \( \exp \) is a group homomorphism we need to check that \( \exp \) maps the identity element in \( \mathbb{R} \) to the identity element in \( \mathbb{C}^\times \) and that \( \exp \) respects the group law. \( \mathbb{R} \) is a group with respect to addition and so identity element in \( \mathbb{R} \) is the number 0. \( \mathbb{R}_{>0} \) is a group with respect to multiplication and the identity element in \( \mathbb{R}_{>0} \) is the number 1. But from the standard properties of the exponential function we have \( e^0 = 1 \) and \( e^{x+y} = e^x e^y \). Hence \( \exp \) is a group homomorphism. To show that \( \exp \) is surjective we need to show that for every \( y \in \mathbb{R}_{>0} \) we can find a number \( x \in \mathbb{R} \) with \( \exp(x) = y \). For this it suffices to choose \( x = \ln(y) \). To show that \( \exp \) is injective we need to show that \( \exp(x_1) = \exp(x_2) \) implies \( x_1 = x_2 \). But \( \exp \) is strictly monotonically increasing so this is clear. Thus \( \exp \) is a group isomorphism.
(b) The fact that \( \exp \) is a homomorphism follows again from the standard properties of the complex exponential function (as in (a)). To show that \( \exp \) is surjective we have to show that for every non-zero complex number \( w \in \mathbb{C}^\times \) we can find a complex number \( z \), so that \( \exp(z) = w \). Write \( w \) in polar form \( w = re^{i\theta} \). Now \( r \) is a positive real number (since \( w \neq 0 \)) and so we can form the real number \( \ln(r) \). Let \( s = |\ln(r)| \). Then \( \ln(r) = s \) if \( \ln(r) \geq 0 \) and \( \ln(r) = -s \) if \( \ln(r) < 0 \). Since \( -1 = e^{i\pi} \) we conclude that \( w = \exp(s + i\theta) \) if \( r \geq 1 \) and \( w = \exp(s + i(\theta + \pi)) \) if \( 0 < r < 1 \). This shows that \( \exp \) is a surjective group homomorphism. The complex exponential \( \exp \) can not be an isomorphism since \( \exp(0) = \exp(2\pi i) = 1 \) but \( 0 \neq 2\pi i \).

(c) The matrix exponential can not be a group homomorphism for \( n \geq 2 \). Indeed consider the \( 2 \times 2 \) matrices

\[
A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Now

\[
\exp(A) = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}, \quad \exp(B) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]

and so

\[
\exp(A) \exp(B) = \begin{pmatrix} e & e \\ 0 & e^{-1} \end{pmatrix}.
\]

On the other hand

\[
A + B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \text{and hence} \quad (A + B)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Using this fact we compute

\[ \exp(A + B) = \sum_{n=0}^{\infty} \frac{1}{n!} (A + B)^n \]

\[ = \sum_{k=0}^{\infty} \frac{1}{(2k)!} I_2 + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (A + B) \]

\[ = \frac{e + e^{-1}}{2} \cdot I_2 + \frac{e - e^{-1}}{2} \cdot (A + B) \]

\[ = \begin{pmatrix} e & \frac{e-e^{-1}}{2} \\ 0 & e^{-1} \end{pmatrix}. \]

Thus \( \exp(A + B) \neq \exp(A) \exp(B) \) and so \( \exp : \mathfrak{gl}(n, \mathbb{C}) \rightarrow GL(n, \mathbb{C}) \) is not a group homomorphism for \( n = 2 \). Finally, for \( n > 2 \) just take the block matrices

\[ S := \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix}, \quad \text{and} \quad T := \begin{pmatrix} B & 0 \\ 0 & I_{n-2} \end{pmatrix}. \]

Now

\[ \exp(S) \exp(T) = \begin{pmatrix} \exp(A) & 0 \\ 0 & e I_{n-2} \end{pmatrix} \cdot \begin{pmatrix} \exp(B) & 0 \\ 0 & e I_{n-2} \end{pmatrix} \]

\[ = \begin{pmatrix} \exp(A) \exp(B) & 0 \\ 0 & e^2 I_{n-2} \end{pmatrix}, \]

and

\[ \exp(S + T) = \begin{pmatrix} \exp(A + B) & 0 \\ 0 & e^2 I_{n-2} \end{pmatrix}. \]

So again \( \exp(S + T) \neq \exp(S) \exp(T) \).
Solution of problem 1.3: If we identify $\mathbb{H}$ with $\mathbb{C}^2$ with the standard unitary product, then $E$ becomes a subspace of $\text{Mat}_{2 \times 2}(\mathbb{C})$ and the coordinates $(x, y, z)$ correspond to the basis of $E$ given by the Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Thus the positive direction of the $y$-axis is the direction $\mathbb{R}_{>0}\sigma_2$ and so the spinor state of an electron with spin projection $1/2$ on the positive direction of the $y$ axis is represented by a norm one eigenvector of $\sigma_2$ corresponding to the positive eigenvalue of $\sigma_2$. The eigenvalues of $\sigma_2$ are $\pm 1$ and the $+1$ eigenspace is spanned by the vector

$$
\begin{pmatrix} -i \\ 1 \end{pmatrix} \in \mathbb{C}^2.
$$

We normalize this vector to have norm one and so

$$
\frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \in \mathbb{C}^2. \quad (1)
$$

represents the normalized state of an electron with spin projection $1/2$ on the positive direction of the $y$ axis.

Similarly the positive direction of the $z$-axis is the direction $\mathbb{R}_{>0}\sigma_3$ and thus the spinor state of an electron with spin projection $1/2$ in this direction will be an eigenvector of $\sigma_3$ corresponding to the positive eigenvalue of $\sigma_3$. Again the eigenvalues of $\sigma_3$ are $\pm 1$ and the $+1$ eigenspace is spanned by the norm one vector

$$
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2. \quad (2)
$$

Now as usual the probability that the electron in the state (1) will pass through a Stern-Gerlah machine filtering only electrons in the state (2) is given by the modulus of the scalar product of two unit norm vectors representing the two states. Thus we get

$$
\text{probability} = \left| \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \right| = \frac{1}{\sqrt{2}}.
$$
Solution of problem 1.4: (a) If \( x, y \in V \) are two vectors we compute

\[
\langle ff^*x, y \rangle = \langle f^*x, f^*y \rangle = \langle x, f^{**}f^*y \rangle = \langle x, ff^*y \rangle.
\]

Thus \( ff^* \) is self adjoint. The same reasoning implies that \( f^*f \) is self-adjoint as well.

(b) For any vector \( x \in V \) we have \( \langle ff^*x, x \rangle = \langle f^*x, f^*y \rangle = \|f^*x\|^2 \geq 0 \). Since \( \|f^*x\|^2 = 0 \) if and only if \( f^*(x) = 0 \), it follows that \( ff^* \) will be positive definite if and only if \( \ker(f^*) \neq 0 \). But the statement that \( \ker(f^*) \neq 0 \) is equivalent to \( f^* : V \to V \) not being an isomorphism, which in turn is equivalent to the adjoint operator \( f \) of \( f^* \) not being an isomorphism.

The prove that \( f^*f \) is positive definite if and only if \( f \) is an isomorphism is completely analogous.

Solution of problem 1.5: (a) Using the multiplication table in \( Q \) we compute

\[
\alpha \bar{\alpha} = (a1 + bi + cj + dk)(a1 - bi - cj - dk) = (a^2 + b^2 + c^2 + d^2)1 + (-ab + ba - cd + dc)i + (-ac + ca - dc + cd)j + (-ad + da - bc + cb)k
\]

\[
= (a^2 + b^2 + c^2 + d^2)1.
\]

(b) If \( \alpha = 0 \) then \( a^2 + b^2 + c^2 + d^2 \neq 0 \) and so

\[
\alpha^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2} \cdot \bar{\alpha}.
\]