(1) Let $G$ be an abelian group and $n$ a fixed positive integer. Show that the subset of all elements in $G$ whose order divides $n$ is a subgroup in $G$. Will this be true if $G$ is non-abelian?

(2) Determine the automorphism group of a cyclic group of order 10.

(3) Let $A$ and $B$ be two groups and let $f, g : A \rightarrow B$ be two group homomorphisms. Consider the map $\varphi : A \rightarrow B$ defined by the formula $\varphi(a) = f(a) \cdot g(a)$ for all $a \in A$. True or false (give a reason or a counter-example):

(a) If $A$ is abelian, the map $\varphi$ is a group homomorphism.
(b) If $B$ is abelian, the map $\varphi$ is a group homomorphism.
(c) If $A = B = S_3$, the map $\varphi$ is a group homomorphism.
(4) Let $G$ be an abelian group of order 12, and let $\varphi : G \rightarrow G$ be the homomorphism given by $\varphi(x) = x^{11}$ for all $x \in G$. Show that $\varphi$ is an isomorphism.

(5) Consider the elements

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \quad y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

in the symmetric group $S_4$ of permutations on four letters. Show that $x$ and $y$ are not conjugate in $S_4$, i.e. show that there is no element $\sigma \in S_4$ satisfying $y = \sigma x \sigma^{-1}$.

(6) Let $G$ be a finite group and let $x, y \in G$ be two elements of order two. Show that the subgroup of $G$ generated by $x$ and $y$ is isomorphic to the dihedral group $D_{2|xy|}$.

(7) Let $G$ be a non-commutative group. Show that $\text{Aut}(G)$ can not be cyclic.

(8) Show that every element in $S_n$ can be written as the product of elements of order two.
(9) Let $S$ be a finite set and let $\sim$ be an equivalence relation on $S$. True or false (give a reason or a counter-example)

(a) The quotient set $S/\sim$ is finite and $|S/\sim| \leq |S|$.
(b) All $\sim$-equivalence classes in $S$ have the same number of elements.
(c) $\sim$ is the relation of equality $=$ if and only if every $\sim$-equivalence class is a singleton.

(10) Let $G$ be an abelian group, and let $H_1, \ldots, H_n < G$ be subgroups. Consider the function

$$\varphi : H_1 \times \cdots \times H_n \to G, \quad \varphi(x_1, \ldots, x_n) := x_1 x_2 \cdots x_n.$$ 

Show that $\varphi$ is an isomorphism if and only if

(i) $\varphi$ is surjective.
(ii) $H_i \cap \langle \cup_{j \neq i} H_j \rangle = \{e\}$ for all $i = 1, \ldots, n$.

(11) Let $\mathcal{C}$ be a category and let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{C}$. Suppose that $g$ is a monomorphism in $\mathcal{C}$ and that $g \circ f$ is an isomorphism. Show that $f$ and $g$ are isomorphisms.

(12) Let $\mathcal{C}$ be a category. Let $I \in \text{Ob} \mathcal{C}$ be an initial object in $\mathcal{C}$, and let $F \in \text{Ob} \mathcal{C}$ be a final object in $\mathcal{C}$. Suppose $\text{Hom}_{\mathcal{C}}(F, I) \neq \emptyset$. Show that $I \cong F$. 
(13) Let \( \text{Rel} \) be the category of relations. Prove that \( \text{Rel} \) has an initial object. Does \( \text{Rel} \) have a final object?

(14) Let \( S \) be a set and let \( C \) be the category associated with the partially ordered set \((\mathcal{P}(S), \subseteq)\). Describe the maximal subgroupoid of \( C \).