Math 503     Fall 2015
Practice Problems for the Midterm

(1) True or false. Give a reason or a counter-example
(a) The map $\sigma : GL_n(\mathbb{R}) \rightarrow \text{Aut}_{\text{Set}}(\text{Mat}_{n \times n}(\mathbb{R}))$ given by $\sigma_P(A) = PAP^t$ defines an action of $GL_n(\mathbb{R})$ on the set of $n \times n$ matrices.
(b) Let $G$ be a group acting on a set $X$. Let $H \subset G$ be the subset $H = \{ g \in G \mid g \cdot x = x \}$. Then $H$ is a normal subgroup of $G$.
(c) If $n \geq 3$, then $S_n$ has trivial center.
(d) If $G$ is a group of odd order and $x \neq e \in G$, then $x$ can not be conjugate to $x^{-1}$.

(2) Determine the automorphism group of a cyclic group of order 10.

(3) Find all finite groups that have exactly two conjugacy classes.
(4) Let $B$ be the group of invertible upper-triangular $n \times n$ matrices. Consider the standard action of $B$ on $\mathbb{R}^n$ and let
\[
a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \vdots \\ 1 \\
\end{pmatrix} \in \mathbb{R}^n.
\]
Describe $\text{Stab}_B(a)$.

(5) Consider the elements
\[
x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \\
\end{pmatrix} \quad y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \\
\end{pmatrix}
\]
in the symmetric group $S_4$ of permutations on four letters. Show that $x$ and $y$ are not conjugate in $S_4$, i.e. show that there is no element $\sigma \in S_4$ satisfying $y = \sigma x \sigma^{-1}$.

(6) Let $G$ be a finite group and let $x, y \in G$ be two elements of order two. Show that the subgroup of $G$ generated by $x$ and $y$ is isomorphic to the dihedral group $D_{2|x||y|}$.

(7) Let $G$ be a non-commutative group. Show that $\text{Aut}(G)$ can not be cyclic.

(8) Suppose a group $G$ acts on a set $X$. Let $x \neq y \in X$ and let $C = \{g \in G \mid g \cdot x = y \}$. Prove that $C$ is a left coset for $\text{Stab}_G(x)$ and a right coset for $\text{Stab}_G(y)$. 
(9) Let $G$ be a group of order $p^k$ with $p$ prime and $1 < k < p$. Prove that $G$ is not simple.

(10) Let $G$ be a group of order $p_1^2p_2^2p_3^2$ with $p_1$, $p_2$, and $p_3$ distinct primes. Suppose that all Sylow subgroups of $G$ are normal. Show that $G$ must be abelian. \textit{Hint:} Show that $G$ must be the product of all its Sylow subgroups.

(11) Prove that if $|G| = 105$ then $G$ has a normal 5-Sylow subgroup and a normal 7-Sylow subgroup.

(12) Find the number of $p$-Sylow subgroups of $A_5$ for $p = 2, 3, 5$.

(13) Let $H \triangleleft G$. Show that $H'$ is a normal subgroup of $G$.

(14) Prove that if $A$ and $B$ are solvable, then $A \times B$ is solvable.

(15) Prove that every group of order $n$ is solvable if (a) $n = 12$, (b) $n = 20$, (c) $n = 100$.

(16) Prove that the dihedral group $D_{16}$ is nilpotent.